CATS competition[1]

University of Kentucky High School Math Day
8 November 2008

Problems and Solutions

1 If $3^{2x-2} = 7$, what is 9^{x-1}?

Solution:

$$9^{x-1} = (3^2)^{x-1} = 3^{2(x-1)} = 3^{2x-2} = 7.$$

2 The sides of a triangle are 4, 9, 11 inches. Find the sides of a similar triangle whose perimeter is 40 inches.

Solution: The sides of a similar triangle are $4k$, $9k$, $11k$ for some k. For the perimeter to be 40, we need

$$4k + 9k + 11k = 24k = 40.$$

Thus $k = 40/24 = 5/3$, and the sides are (in inches)

$$4k = 20/3, \quad 9k = 15, \quad 11k = 55/3.$$

3 If $x = 3$ and $x = 1/6$ are solutions of the equation

$$ax^2 + bx + 1 = 0,$$

find a and b.

Solution: Since $x = 3$ is a solution, we have:

$$9a + 3b + 1 = 0. \tag{3.1}$$

Similarly, since $x = 1/6$ is a solution, we have:

$$\frac{a}{36} + \frac{b}{6} + 1 = 0,$$

or, multiplying the equation by 36:

$$a + 6b + 36 = 0. \tag{3.2}$$

To find a, subtract twice (3.1) from (3.2):

$$-17a + 34 = 0,$$

or $a = 2$. Then equation (3.2) becomes

$$6b + 38 = 0,$$

and thus $b = -19/3$.

[1]CATS stands for CATS Are Top Solvers.

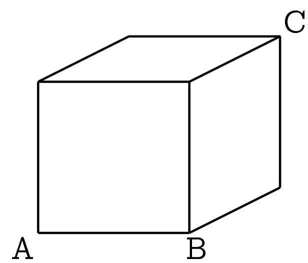
4 A car goes 51 miles per hour for 2 hours and then 66 miles per hour for 3 hours. What is the average speed of the car for the 5 hour trip?

SOLUTION: The total distance traveled by the car is

$$(2 \times 51) + (3 \times 66) = 300 \quad \text{(miles)}.$$

Thus the average speed was $300/5 = 60$ mph.

5 In the cube below, $|AB| = 1$. Find $|AC|$.



SOLUTION: AC is the hypotenuse of the right triangle ABC. We have $|BC| = \sqrt{1^2 + 1^2} = \sqrt{2}$, and $|AB| = 1$. Thus

$$|AC| = \sqrt{|AB|^2 + |BC|^2} = \sqrt{1+2} = \sqrt{3}.$$

6 If N^2 is a divisor of $8! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8$, what is the largest possible integer value of N?

SOLUTION: We have

$$8! = 2 \cdot 3 \cdot 2^2 \cdot 5 \cdot (2 \cdot 3) \cdot 7 \cdot 2^3 = 2^7 \cdot 3^2 \cdot 5 \cdot 7.$$

It is now clear that the largest square N^2 that divides $8!$ is $2^6 \cdot 3^2 = 24^2$. Thus $N = 24$.

7 The arithmetic mean of three numbers x, y, z is 24. The arithmetic mean of $x, 2y, z-7$ is 35. What is the arithmetic mean of x and z?

SOLUTION: The first condition says that

$$x + y + z = 3 \cdot 24 = 72. \tag{7.1}$$

The second says that

$$x + 2y + (z - 7) = 3 \cdot 35 = 105,$$

or,

$$x + 2y + z = 112. \tag{7.2}$$

Subtracting (7.2) from twice (7.1), we have:

$$x + z = 32.$$

Thus the average of x and z is 16.

8 Cucumbers are normally 96% water. 100 pounds of cucumbers were left in the sun and are now 92% water. What is the weight of the cucumbers now?

SOLUTION: The "non-water" in cucumbers is normally 4%. The *same* non-water is now 8%. Thus the total weight is now half of what it was, i.e. 50 pounds.

9 Find the least common multiple of 77 and 21.

SOLUTION: We have

$$77 = 7 \cdot 11, \quad 21 = 3 \cdot 7.$$

Thus the least common multiple is

$$3 \cdot 7 \cdot 11 = 231.$$

10 What is the last digit of 9^{2008}?

SOLUTION: We have

$$9^1 = 9, \quad 9^2 = 81, \quad 9^3 = 729, \text{ etc.}$$

Thus the odd powers of 9 end with 9 and the even powers end with 1. Therefore 9^{2008} ends with 1.

11 What are the last two digits of 7^{999}?

SOLUTION: We have

$$7^1 = 7, \quad 7^2 = 49, \quad 7^3 = 343, \quad 7^4 = 2401.$$

It follows that

$$7^4, 7^8, 7^{12}, \ldots$$

end with 01. In particular,

$$7^{996} = 7^{4 \cdot 249}$$

ends with 01. Since

$$7^{999} = 7^{996} \times 343,$$

we see that 7^{999} ends with 43.

12 If we make one straight cut through a pizza, we obtain two pieces of pizza. If we make another cut, we obtain three or four pieces of pizza depending on how we cut. What is the largest number of pieces we can obtain after five cuts?

SOLUTION: Extending each cut indefinitely, we get straight lines in the plane. The second cut (line) intersects at most one other line (the first cut). The third line intersects at most two lines, and so on. Thus the nth line intersects at most $n-1$ lines that divide the nth line into n segments. Each of these segments adds a new piece of pizza to the pieces formed by the previous $n-1$ cuts. Thus adding an nth cut increases the number of pieces by at most n. We now see that five cuts result in at most $2+2+3+4+5 = 16$ pieces of pizza.

13 Find the sum

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots + \frac{1}{199 \cdot 201}.$$

Solution: We have

$$\begin{aligned}
&\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots+\frac{1}{199\cdot 201}\\
&\quad=\frac{1}{2}\left(1-\frac{1}{3}\right)+\frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}\right)+\frac{1}{2}\left(\frac{1}{5}-\frac{1}{7}\right)+\cdots+\frac{1}{2}\left(\frac{1}{199}-\frac{1}{201}\right)\\
&\quad=\frac{1}{2}\left(1-\frac{1}{3}+\frac{1}{3}-\frac{1}{5}+\frac{1}{5}-\frac{1}{7}+\cdots+\frac{1}{199}-\frac{1}{201}\right)\\
&\quad=\frac{1}{2}\left(1-\frac{1}{201}\right)=\frac{100}{201}.
\end{aligned}$$

14 Factor x^4+4 as a product of two polynomials.

Solution: Here is how one can factor x^4+4 as a product of polynomials with real coefficients. We have:

$$\begin{aligned}
x^4+4 &= (x^4+4x^2+4)-4x^2\\
&= (x^2+2)^2-(2x)^2\\
&= (x^2+2+2x)(x^2+2-2x)\\
&= (x^2+2x+2)(x^2-2x+2).
\end{aligned}$$

Over the complex numbers, factoring is even easier. For example:

$$x^4+4=(x^2)^2-(2i)^2=(x^2+2i)(x^2-2i).$$

15 Suppose that the area of a rectangle is 20 square units and its perimeter is 20 units. The length of the diagonal is $\sqrt{n}$ where n is a whole number. Find n.

Solution: Let a and b be the sides of the rectangle. Then

$$ab=20, \quad \text{and} \quad 2(a+b)=20 \quad \text{i.e.} \quad a+b=10.$$

Thus

$$n=a^2+b^2=(a+b)^2-2ab=10^2-2\cdot 20=60.$$

16 License plates in Mathfunland use 4 digit numbers. Only the digits from 1 to 9 are used (0 not allowed), and all four digits must be different. How many different license plates are possible in Mathfunland?

Solution: There are 9 possible choices for the first digit. Once the first digit is chosen, there are 8 possible choices for the second digit. Once the first two digits are chosen, there are 7 possible choices for the third digit. Finally, once the first three digits are chosen, there are 6 possible choices for the last digit. Thus the total number of possible choices for the license plate numbers is $9\cdot 8\cdot 7\cdot 6=3024$.

17 If we expand the expression $(x-3y)^2=x^2-6xy+9y^2$, the coefficients are $1, -6$ and 9 and the sum of these coefficients is 4. Find the sum of the coefficients of $(x-2y)^{10}$.

SOLUTION: To calculate the sum of the coefficients of a polynomial in x and y, it is enough to set $x = y = 1$. Thus for the polynomial $(x-3y)^2 = x^2 - 6xy + 9y^2$, the sum is

$$1^2 - 6 \cdot 1 \cdot 1 + 9 \cdot 1^2 = (1-3)^2 = 4.$$

For $(x-2y)^{10}$, the sum is $(1-2)^{10} = 1$.

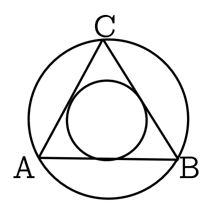
18 Six distinct straight lines are drawn on the plane. What is the largest possible number of intersections these lines can have?

SOLUTION: The largest number of intersections occurs when no two lines are parallel. In this case, there are as many intersections as there are pairs of lines. For 6 lines, there are 15 pairs possible. This is easy to see by enumeration: if we label the lines 1, 2, 3, etc., the possible pairs are:

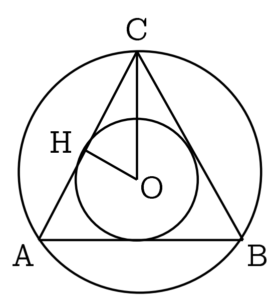
$$12, 13, 14, 15, 16, 23, 24, 25, 26, 34, 35, 36, 45, 46, 56.$$

Thus the largest possible number of intersections for 6 lines is 15.

19 $\triangle ABC$ is equilateral. If the area of the inscribed circle is 1, what is the area of the circumscribed circle?



SOLUTION: Let O be the center of the two circles and OH be perpendicular to AC, as on the figure below:



Then OH and OC are radii of the two circles. Moreover, it is easy to see that $\triangle COH$ is a 30-60-90 triangle. In particular, $|OC| = 2\,|OH|$. Thus the radius of the large circle is twice the radius of the small circle. Therefore the area of the large circle is four times the area of the small circle. The answer is: 4.

20 In a game, players can score either 7 or 3 points each time. If a player scores 82 points, what are the possible numbers of 3 point goals that were scored?

SOLUTION: Let m and n be the numbers of 3 point and 7 point goals scored respectively. Then $3m + 7n = 82$. Thus

$$m = \frac{82 - 7n}{3}.$$

It is easy to see that the values of n for which $82 - 7n$ is positive and divisible by 3 are $n = 1, 4, 7, 10$. The corresponding values of m are $25, 18, 11, 4$.

21 Find the sum of the even numbers from 2 to 2008, $2 + 4 + 6 + \cdots + 2008$.

SOLUTION: We have

$$\begin{aligned} 2 + 4 + 6 + \cdots + 2008 &= (1 + 2 + 3 + \cdots + 1004) + (1004 + 1003 + 1002 + \cdots + 1) \\ &= (1 + 1004) + (2 + 1003) + (3 + 1002) + \cdots (1004 + 1) \\ &= 1005 + 1005 + 1005 + \cdots + 1005 \quad \text{(1004 terms)} \\ &= 1004 \times 1005 = 1009020. \end{aligned}$$

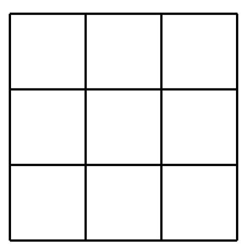
22 If $n!$ stands for the product $1 \cdot 2 \cdot 3 \cdot \cdots (n-1)n$, evaluate

$$\frac{10!}{7!\ 6!}.$$

SOLUTION: We have $10! = 7! \times 8 \times 9 \times 10) = 7! \times 720 = 7!\ 6!$. Thus

$$\frac{10!}{7!\ 6!} = 1.$$

23 How many different squares are drawn in the following grid:

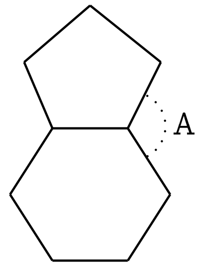


SOLUTION: There are nine 1×1 squares, four 2×2 squares, and one 3×3 square, for a total of 14 squares.

24 The number $n!$ is the product of the whole numbers from 1 to n. Thus $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$. The number 120 ends with one zero. How many consecutive zeros are there at the end of $100!$?

SOLUTION: Among the numbers $1, 2, 3, \ldots, 100$ there are twenty that are divisible by 5 and among these twenty, four are divisible by 25. Thus $100!$ is divisible by 5^{24} but not by 5^{25}. Since there are fifty even numbers among $1, 2, 3, \ldots, 100$, we also know that $100!$ is divisible by 2^{24}. It follows that $100!$ is divisible by 10^{24} but not by 10^{25}. Therefore, $100!$ ends with 24 zeroes.

25 The sketch shows a regular pentagon and a regular hexagon that share a side. Find the measure of the angle A in degrees.



Solution: The measure of each angle of a regular pentagon is 108 degrees. The measure of each angle of a regular hexagon is 120 degrees. Thus the measure of the angle A is $360 - 108 - 120 = 132$ degrees.

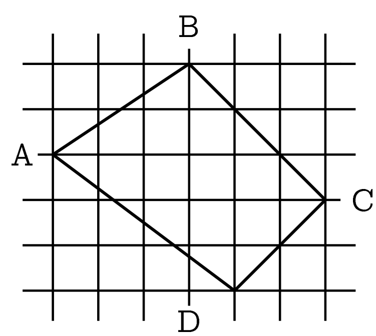
26 Factor $9991 = A \cdot B$, where A and B are whole numbers with $1 < A < B$. Find A.

Solution: We have

$$9991 = 10000 - 9 = 100^2 - 3^2 = (100 - 3)(100 + 3) = 97 \cdot 103.$$

Thus $A = 97$.

27 The little squares in the grid below are 1×1. Find the area of the quadrilateral $ABCD$.



Solution: The quadrilateral is naturally inscribed in a 5×6 rectangle with sides along the lines of the grid. Its area (30 square units) is composed of the area of the quadrilateral and the combined areas of the four right triangles whose hypotenuses are AB, BC, CD, and DA. The areas of these triangles are $3, 4.5, 2$, and 6 respectively. Thus the area of the quadrilateral is

$$30 - 3 - 4.5 - 2 - 6 = 14.5.$$

28 If $n = 10101$ in base 2, give the number n in base 10.

Solution:

$$n = 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 16 + 0 + 4 + 0 + 1 = 21.$$

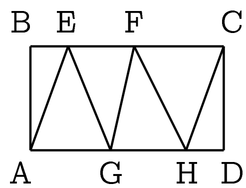
29 We have a two digit prime number and when the digits are interchanged the value of the number decreases by 36. What is the number?

SOLUTION: Let a and b be the digits of our number, i.e. the number is $10a + b$. The condition on the digits is that

$$(10a + b) - (10b + a) = 36,$$

i.e. $9a - 9b = 36$ or $a - b = 4$. Thus the only possibilities are $40, 51, 62, 73, 84, 95$. Of these numbers, only 73 is prime. Thus the answer is 73.

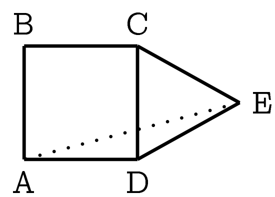
30 The area of the rectangle $ABCD$ is 6 square units. What is the combined area of the three triangles $\triangle AEG$, $\triangle GFH$, and $\triangle HCD$?



SOLUTION: The three triangles have the same height $|AB|$. Therefore their combined area is

$$\frac{|AG| \cdot |AB|}{2} + \frac{|GH| \cdot |AB|}{2} + \frac{|HD| \cdot |AB|}{2} = \frac{|AD| \cdot |AB|}{2} = \frac{6}{2} = 3.$$

31 $ABCD$ is a square, the triangle $\triangle CDE$ is equilateral. Find the measure of the angle $\angle DAE$ (in degrees).



SOLUTION: The measure of $\angle ADE$ is $90 + 60 = 150$ degrees. The triangle $\triangle ADE$ is isosceles because $|DA| = |DC| = |DE|$. Therefore the measure of $\angle DAE$ is

$$\frac{1}{2}(180 - 150) = 15 \quad \text{(degrees.)}$$

32 List all the integers $n \geq 0$ for which $2^n + 1$ is the square of an integer.

SOLUTION: If $2^n + 1 = m^2$, we have $2^n = m^2 - 1 = (m-1)(m+1)$. Thus $m - 1$ and $m + 1$ must be consecutive powers of 2. Clearly, the only possibility is $m = 3$, i.e. $2^n = 8$ and $n = 3$.

33 If $x + y = 10$ and $xy = 5$, what is $x^3 + y^3$?

SOLUTION: Since $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 = x^3 + y^3 + 3xy(x+y)$, we have

$$x^3 + y^3 = (x+y)^3 - 3xy(x+y) = 10^3 - 3 \cdot 5 \cdot 10 = 850.$$