Spanning Tree Formulas for Abelian Covers of Multigraphs and Artin-Ihara L-functions

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Chapter 1

Introduction and Motivation

In this thesis, we consider Artin-Ihara L-functions attached to Galois coverings of finite connected multigraphs. In this setting, we will study analogies between Galois coverings of multigraphs and Galois extensions of number fields. In particular, in Chapter 3, we prove a spanning tree formula for Galois covers of finite connected multigraphs where $Aut(Y/X)$ is an elementary abelian 2-group. When $Aut(Y/X) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, our result can be viewed as an analogue of Kuroda's class number formula for biquadratic extensions. See [3].

1.1 Preliminaries

A **multigraph** X consists of a set V_X, and a multiset E_X of pairs of elements of V_X. We say a multigraph is **finite** if V_X and E_X are finite. We interpret the elements of V_X as **vertices** and the elements of E_X as **edges** connecting vertices. We point out that a multigraph can have multiple edges connecting any two vertices, called **multiple edges**, and edges connected to the same vertex, called **loops**. An example of a multigraph is given in Figure 1.2. A **(simple) graph** is a multigraph that have neither multiple edges nor loops. A **subgraph** is a multigraph X' such that $V_{X'} \subseteq V_X$ and $E_{X'} \subseteq E_X$.

A vertex v is **incident** to the edge e if $v \in e$. The **degree** of a vertex v, written as $deg(v)$ or $d(v)$ is defined as the number of edges incident to an edge, where loops are counted twice. For example, in Figure 1.1, $d(v_2) = 3$. Also, in Figure 1.2, $d(c) = 6$.

A directed edge is any edge $e = (a, b)$, that has a specified direction given

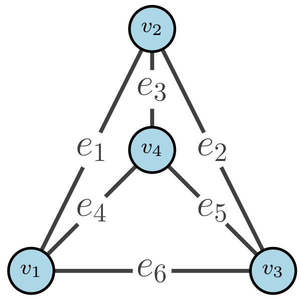

Figure 1.1: The graph K_4

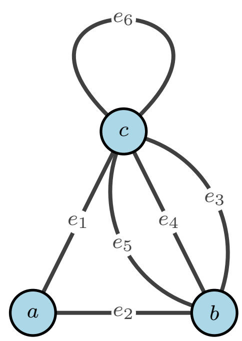

Figure 1.2: An example of a multigraph on 3 vertices.

to it. We then define $e^{-1} = (b, a)$ as the edge with opposite direction as e. The **initial vertex** of $e = (a, b)$ is a and the **terminal vertex** b. A **path** P is a sequence of directed edges $e_1 e_2 \ldots e_n$ such that the terminal vertex of e_i is the initial vertex of e_{i+1}. Given two paths P_1 and P_2 where the terminal vertex of P_1 is the initial vertex of P_2, we define $P_1 \cdot P_2$ to be the path P_1 followed by P_2. The path P is called **closed** if the terminal vertex of e_n is the initial vertex of e_1. A **cycle** is a closed path P such that no edge is repeated independently of the orientation, and such that the terminal vertices of e_i for $i = 1, \ldots, n$ are distinct. A multigraph is **connected** if there exists a path between any two vertices. *Throughout this thesis a multigraph will always be finite and connected unless otherwise stated.*

A **tree** is a graph that has no cycles. A **spanning tree** $T \subseteq X$ is a subgraph of X such that $V_T = V_X$ and T is a tree. The **number of**

spanning trees of a multigraph X is denoted by κ_X.

From now on, we label the vertices $v_1, v_2, \ldots, v_n$ and in doing so we introduce an arbitrary ordering of the vertices. This ordering will not affect any of the forthcoming results. The **adjacency matrix** A of a multigraph X is defined as the $n \times n$ matrix given by

$$A_{i,j} = \begin{cases} \text{number of edges connecting the vertex } v_i \text{ to } v_j, & \text{if } i \neq j; \\ 2 \times \text{number of loops at } v_i, & \text{if } i = j. \end{cases}$$

The **degree matrix** D is the $n \times n$ diagonal matrix given by

$$D_{i,j} = \begin{cases} d(v_i), & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$$

The **Laplacian matrix** L is defined to be $D - A$.

Example 1.1.1. Let X be K_4, the graph with four vertices, where there is a unique edge between every pair of distinct vertices (see Figure 1.1). The adjacency matrix of X is

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix},$$

the degree matrix of X is

$$D = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix},$$

and the Laplacian matrix is

$$L = D - A = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}.$$

For Theorem 1.1.2 below, we would like to remind the reader about the **adjoint** or **adjugate** of a matrix. Let M be a matrix and let $B_{i,j}$ be the determinant of the (i, j) minor of M. Then the adjugate of M is $adj(M) = ((-1)^{i+j} B_{i,j})^t$.

Theorem 1.1.2. *(Matrix-Tree Theorem) Let X be a multigraph, and L the Laplacian matrix. Then,*

$$adj(L) = \kappa_X \cdot J,$$

where J is the matrix whose entries are all 1's.

Proof. See Theorem 6.3 in [1]. The proof by Biggs is only for graphs, but can be extended to multigraphs. □

Example 1.1.3. Let X be K_4. Then the Laplacian matrix is given in Example 1.1.1. One calculates that

$$adj(L) = 16 \cdot J,$$

and therefore $\kappa_X = 16$ as one can also check directly by listing all the spanning trees.

1.2 The Ihara Zeta Function

Let X be a multigraph. A closed path $P = e_1 e_2 \ldots e_m$ has a **backtrack** if $e_{j+1} = e_j^{-1}$ for some j. The path P has a **tail** if $e_m = e_1^{-1}$. The **length** of a path $P = e_1 e_2 \ldots e_m$ is $\mathcal{V}(P) = m$. A **prime path** in a multigraph is a closed path that has no backtrack, no tail, and is not of the form C^f for some other closed path C. We define an equivalence relation $\sim$ on the set of prime paths as follows. We say that $P_1 \sim P_2$ if the two prime paths only differ by the choice of a starting vertex. The equivalence classes will be called **primes**. We denote primes by $\mathfrak{p}$.

We also have the fundamental group of a graph X which is a free group of rank r_X where

$$r_X = |E_X| - |V_X| + 1.$$

From now on, we assume that our multigraphs have no degree one vertices.

Definition 1.2.1. The **Ihara Zeta Function** is defined to be the (usually) infinite formal product

$$\zeta_X(u) = \prod_{\mathfrak{p}} (1 - u^{\mathcal{V}(\mathfrak{p})})^{-1},$$

where the product is over all primes $\mathfrak{p}$ of X. Recall that $\mathcal{V}(\mathfrak{p})$ is the length of the prime $\mathfrak{p}$. One can show that this infinite product converges if u is a complex number such that $|u|$ is small enough.

We point out that this product is usually infinite unless the graph is a cycle graph, in which case there are only two primes.

Theorem 1.2.2. *(Three Term Determinant Formula) Let A and D be the adjacency and degree matrix of X respectively. Let r_X be the rank of the fundamental group of X, then*

$$\zeta_X(u)^{-1} = (1-u^2)^{r_X-1} det(I - Au + (D-I)u^2).$$

Proof. See page 89 of [5]. □

Example 1.2.3. Let $X = K_4$, the graph given in Figure 1.1. The adjacency matrix A and the degree matrix D are given in Example 1.1.1. We apply Theorem 1.2.2 which gives the following:

$$\begin{aligned}\zeta_X(u)^{-1} &= (1-u^2)^2 det(I - Au + (D-I)u^2) \\ &= (u+1)^2(u-1)^3(2u-1)(1+u+2u^2)^3\end{aligned}$$

Theorem 1.2.5 below is analogous to the analytic class number formula in algebraic number theory. This theorem is the starting point of our research. But first, we need a lemma before we begin proving Theorem 1.2.5.

Lemma 1.2.4. *(Handshake Lemma) If X is a multigraph then*

$$\sum_{v \in V_X} d(v) = 2|E_X|$$

Proof. The degree counts the number of edges incident to a vertex. But if we sum all the degrees it counts every edge twice since every edge is incident to two vertices and one loop contributes 2 to the degree of that vertex. Therefore the result is proven. □

The following result is an exercise on page 78 of [5], but we present a proof for the reader's convenience.

Theorem 1.2.5. *(Analytic Spanning Tree Formula) The Ihara Zeta Function satisfies*

$$\frac{d^r}{du^r}\zeta_X(u)^{-1}|_{u=1} = (-1)^{r-1}2^r r!(r-1)\kappa_X,$$

where $r = |E_X| - |V_X| + 1$.

Proof. As stated in Theorem 1.2.2

$$\zeta_X(u)^{-1} = (1-u^2)^{r-1}\det(I - Au + (D-I)u^2).$$

Let $g(u) = (1-u^2)^{r-1}$ and $h(u) = det(I - Au + (D-I)u^2)$. By the Leibniz formula for the generalized product rule,

$$\frac{d^r}{du^r}(h(u)g(u)) = \sum_{i=0}^{r} \binom{r}{i} g^{(r-i)}(u)h^{(i)}(u). \tag{1.1}$$

The function g has a zero of order $r-1$ at 1, so when we evaluate Equation (1.1) at 1, we get

$$\frac{d^r}{du^r}\zeta_X(u)^{-1}|_{u=1} = rg^{(r-1)}(1)h'(1) + g^r(1)h(1).$$

We know that $h(1) = det(I - A + (D-I)) = det(D-A) = det(L)$. One sees that 0 is an eigenvalue of L, which makes the $det(L) = 0$. So,

$$\frac{d^r}{du^r}\zeta_X(u)^{-1}|_{u=1} = rg^{(r-1)}(1)h'(1). \tag{1.2}$$

Now we consider the $rg^{(r-1)}(1)$ part of Equation (1.2). As stated above, g has a zero of order $r-1$ at 1, so $rg^{r-1}(1) = (-1)^{r-1}2^{r-1}r!$. It is left to show that $h'(1) = 2(r-1)\kappa_X$. To do this, we will use Jacobi's derivative formula for the determinant of a matrix function. It states that

$$\frac{d}{dt}(A(t)) = Tr\left(adj(A(t))\frac{dA(t)}{dt}\right).$$

Therefore

$$h'(u) = Tr(adj(I - Au + (D-I)u^2)(-A + 2(D-I)u)).$$

Evaluating $h'(1)$ we get

$$h'(1) = Tr(adj(L)(-A + 2D - 2I)).$$

By Theorem 1.1.2, we have that $adj(L) = \kappa_X \cdot J$. This means

$$\begin{aligned} Tr(adj(L)(-A + 2D - 2I)) &= Tr(\kappa_X \cdot J(-A + 2D - 2I)) \\ &= \kappa_X Tr(J(-A + 2D - 2I)). \end{aligned}$$

Since the trace is a linear operator, one can turn this expression into

$$\kappa_X(Tr(JL) + Tr(JD) - Tr(2J)).$$

We will take each of these term by term. The first term $Tr(JL) = 0$, since the sum of any row of L is 0, therefore multiplying by J, each diagonal entry is 0. Next, $Tr(JD) = 2|E_X|$. This is because $Tr(JM) = Tr(M)$ whenever M is diagonal so $Tr(JD) = Tr(D) = \sum_{v \in V_X} d(v)$. By Lemma 1.2.4, this sum is $2|E_X|$. And lastly, we show $Tr(2J) = 2|V_X|$. Since J is a $|V_X| \times |V_X|$ matrix, $Tr(2J) = 2Tr(J) = 2|V_X|$. So,

$$h'(1) = \kappa_X(2|E_X| - 2|V_X|)$$

which is $2(r-1)\kappa_X$ as we wanted to show. □

Chapter 2

Galois Covers of Graphs

In this section we study covering spaces of graphs, and L-functions associated to Galois covers.

Definition 2.0.1. (Galois Cover) Let $p : Y \to X$ be an unramified cover of multigraphs as defined on page 20 of [5]. Such a cover is a d to 1 map for some positive integer d. We say that Y is a Galois (or normal) cover of X if there are exactly d automorphisms σ of Y such that $p \circ \sigma = p$. We denote the set of all such automorphisms by $Aut(Y/X)$.

A convenient way of constructing Galois covers is to use the notion of voltages.

2.1 Voltages

Let X be a multigraph and let G be a finite group. We arbitrarily direct the edges of X, and we denote the set of directed edges by $\overrightarrow{E}$. Following [2], we call a function $\alpha : \overrightarrow{E} \to G$ a **voltage assignment** on X. Using this voltage, we can construct a multigraph Y, sometimes called the **derived multigraph**, which is a Galois cover of X with Galois group G as follows.

The vertex set of Y is $V_Y = V_X \times G$ and the edge set is $E_Y = E_X \times G$. The edge $(e, g) \in E_Y$ goes from (u, g) to $(v, g \cdot \alpha(e))$ if $e = (u, v)$. The natural projection map $p : Y \to X$ is then simply the map satisfying $p((v, g)) = v$ and $p((e, g)) = e$ for every $g \in G$.

As we pointed out earlier, we always assume that our multigraphs are connected, but the derived multigraph might not be. If we assume that Y is also connected, then we have the following theorem.

Theorem 2.1.1. *Let Y be the derived multigraph constructed from a voltage assignment $\alpha : \overrightarrow{E_X} \to G$ for some finite group G. If Y is also connected, then the group of automorphisms $Aut(Y/X)$ is isomorphic to G*

Proof. Let $g \in G$ and let $p : Y \to X$ be a projection map from Y to X. For each g there is an associated graph automorphism ϕ_g defined by $\phi_g((v,h)) = (v, gh)$ and for edges $\phi_g((e,h)) = (e, gh)$. We will show that $\phi_g \in Aut(Y/X)$. Let $(e,h) = (((v_1,h),(v_2,h\cdot\alpha(e))))$, then

$$\phi_g((e,h)) = (e,gh) = ((v_1,gh),(v_2,g(h\cdot\alpha(e)))) = ((v_1,gh),(v_2,(gh)\cdot\alpha(e))).$$

This shows that $\phi_g \in Aut(Y/X)$. Let $i : G \to Aut(Y/X)$ such that $i(g) = \phi_g$. We will show that i is an isomorphism of groups. First, we show i is a homomorphism. Let $a, b \in G$, then $i(ab) = \phi_{ab} = \phi_a \circ \phi_b = i(a)i(b)$. Now we need i to be injective. Assume that $i(g_1) = i(g_2)$. Then $\phi_{g_1} = \phi_{g_2}$. Select any vertex $(v, g) \in V_Y$ and we get

$$\phi_{g_1}((v,g)) = (v, g_1 \cdot g) = (v, g_2 \cdot g) = \phi_{g_2}((v,g)).$$

By cancellation $g_1 = g_2$. We now need to show that i is surjective. If $\phi \in Aut(Y/X)$ and $(v,h) \in V_Y$ then $\phi((v,h)) = (v', h')$. But $p \circ \phi = p$, so $v = v'$. Thus $\phi((v,h)) = (v,h')$ for some $h' \in G$. Let $g = h'h^{-1}$. Then $\phi \circ \phi_g^{-1} \in Aut(Y/X)$ and has a fixed point, namely (v,h'). This implies that $\phi = \phi_g$ by a standard lemma in covering space theory. (See Lemma 2.2.1 on page 30 of [4].)

□

Theorem 2.1.1 allows us to use voltages to construct Galois covers of multigraphs.

Example 2.1.2. We can construct the cover shown in Figure 2.2 with base graph K_4 (Figure 1.1) by using the following voltage. First we choose an arbitrary direction on the edges of K_4 as in Figure 2.1. Let $\alpha : \overrightarrow{E_{K_4}} \to \mathbb{Z}/2\mathbb{Z}$ be the voltage assignment:

$$\alpha(e_i) = \bar{1} \text{ for every } i = 1, \ldots, 6.$$

Evaluating this function and constructing the voltage graph gives us C_3, the cube as in Figure 2.2.

Our projection map $p : C_3 \to K_4$ sends

$$p(v_i) = p(v_i') = v_i \text{ for all } i = 1, 2, 3, 4,$$

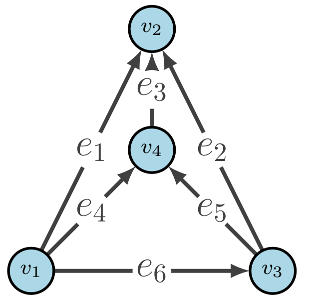

Figure 2.1: The graph K_4 with a choice of directions.

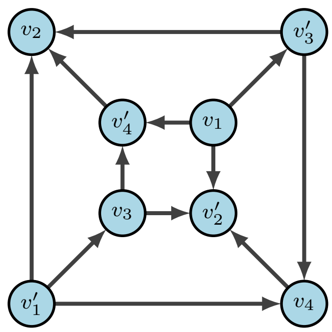

Figure 2.2: The cube forms a $\mathbb{Z}/2\mathbb{Z}$-cover of K_4

where $v_i = (v_i, \bar{0})$ and $v_i' = (v_i, \bar{1})$. We have two automorphisms σ_1,$\sigma_2 \in Aut(Y/X)$. The automorphism σ_1 is the identity and σ_2 satisfies

$$\sigma_2(v_i) = v_i' \text{ and } \sigma_2(v_i') = v_i.$$

2.2 Artin-Ihara L-functions

From now on, G will always stand for a finite abelian group unless otherwise stated.

2.2.1 Characters of finite abelian groups

We begin with a short discussion of characters of finite abelian groups. Let G be a finite abelian group, and let $\chi : G \to \mathbb{C}^\times$ be a function. We call χ a character of G if this χ is a group homomorphism.

Lemma 2.2.1. *The values of a character χ of a finite abelian group G are dth roots of unity, where $d = |G|$.*

Proof. Let $d = |G|$. Then $g^d = e_G$ for all $g \in G$. Then $1 = \chi(e_G) = \chi(g^d) = \chi(g)^d$, since χ is a group homomorphism. So $\chi(g)$ must be a dth root of unity. □

We have the trivial character χ_1 such that $\chi_1(g) = 1$ for every $g \in G$. We also introduce the conjugate character $\overline{\chi}$ of χ which is defined to be $\overline{\chi}(g) = \overline{\chi(g)}$ for all $g \in G$. It is well-known that there are precisely $|G|$ different characters of G.

Example 2.2.2. Let $G = \mathbb{Z}/4\mathbb{Z}$. There are four characters listed in the table below:

	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$
χ_1	1	1	1	1
χ_2	1	-1	1	-1
χ_3	1	i	-1	$-i$
χ_4	1	$-i$	-1	i

2.2.2 The Frobenius Automorphism

We start with the following important definition.

Definition 2.2.3. Let Y/X be a cover with projection map p and $[P^*]$ a prime in Y. If $p(P^*) = P^f$ for some prime $[P]$ in X and some integer f, then we say that $[P^*]$ is a prime above $[P]$ and the residual degree of $[P^*]$ is f.

We will use this definition to define the Frobenius automorphism associated to a prime of Y lying above a prime of X. Before doing so, we need the following theorem.

Theorem 2.2.4. *Let Y be a cover (not necessarily Galois) of a graph X with projection map p. Let C be a closed path in X. Then there is a unique lift C^* in Y such that $p(C^*) = C$ once the initial vertex of C^* is fixed*

Proof. Fix the initial vertex of C^* to be v_1'. Since C is a closed path it is of the form $e_1e_2 \ldots e_n$ such that the initial vertex of e_1 is the terminal vertex of e_n. Choose the edge e of C^* such that $p(e) = e_1$. You can now continue this process, finding the correct edge that projects down to the edge of C that comes next. That path that is created is your unique lift. □

If the cover is a d to 1 map, then as you vary the initial vertex there will be d different lifts of C.

Definition 2.2.5. Let Y/X be a Galois cover with projection p. Let $\mathfrak{P} = [P^*]$ be a prime lying above $\mathfrak{p} = [P]$ and let us fix an initial vertex w of P^*. Furthermore, let $v = p(w)$. By Theorem 2.2.4, we have a unique lift of P to Y that starts at w and ends at w' for another vertex w' satisfying $p(w') = v$. The **Frobenius Automorphism** is the unique automorphism $g \in Aut(Y/X)$ satisfying $g \ \cdot w = w'$ and is denoted by

$$[Y/X, \mathfrak{P}].$$

By (3) of Proposition 16.5 on page 137 of [5], if the automorphism group is abelian then the Frobenius automorphism depends only on the prime lying below. In this case it will be denoted by $[Y/X, \mathfrak{p}]$.

2.2.3 Artin-Ihara L-functions

We can now define the analogue of Artin L-functions for Galois covers of multigraphs.

Definition 2.2.6. Let Y/X be a Galois cover of multigraphs and let $G = Aut(Y/X)$ be abelian. Then one defines

$$L(u, \chi) = \prod_{\mathfrak{p}} (1 - \chi([Y/X, \mathfrak{p}])u^{\mathcal{V}(\mathfrak{p})})^{-1},$$

where the product is over all primes $\mathfrak{p}$ of X.

It is simple to see that $L(u, \chi_1, Y/X) = \zeta_X(u)$. We now introduce definitions that will allow us to formulate a three term determinant formula for L-function, similar to Theorem 1.2.2. Let us fix a spanning tree T of X. It has a lifting to a subtree T_Y, which we also fix, on which the covering map gives an isomorphism of graphs between T and T_Y. For each $i = 1, \ldots, n$, there exists a unique vertex w_i of T_Y lying above v_i which we will denote by w_i $(i = 1, \ldots, n)$.

Definition 2.2.7. Let Y/X be an abelian cover of multigraphs with automorphism group G. For $\sigma \in G$, we define the matrix $A(\sigma)$ to be the $n \times n$ matrix $A(\sigma) = (a_{ij}(\sigma))$ defined via

$$a_{ij}(\sigma) = \begin{cases} \text{Twice the number of loops at the vertex } w_i, & \text{if } i = j \text{ and } \sigma = 1; \\ \text{The number of edges connecting } w_i \text{ to } w_j^{\sigma}, & \text{otherwise.} \end{cases}$$

Definition 2.2.8. Let χ be a character of $Aut(Y/X)$ and $A(\sigma)$ be given by Definition 2.2.7, then one defines the **Artinized adjacency matrix** to be

$$A_\chi = \sum_{\sigma \in G} A(\sigma) \cdot \chi(\sigma).$$

Theorem 2.2.9. *With the notation of Definition 2.2.8, we have*

$$L(u, \chi, Y/X)^{-1} = (1-u^2)^{r_X - 1} det(I - A_\chi u + (D-I)u^2)$$

Proof. See Theorem 18.15 on page 156 of [5] □

This looks very similar to Theorem 1.2.2, but the adjacency matrix is twisted by a character χ.

Example 2.2.10. We now calculate the L-function associated to the unique non-trivial character χ of the $\mathbb{Z}/2\mathbb{Z}$-cover of Example 2.1.2. We have

$$A(1) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$A(\sigma) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Then

$$A_\chi = A(1) - A(\sigma) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & -1 \\ 1 & -1 & -1 & 0 \end{bmatrix}$$

Then

$$L(u, \chi, Y/X)^{-1} = (u-1)^2(u+1)^3(2u+1)(2u^2-u+1)^3.$$

Theorem 2.2.11. *(Factorization of the Ihara Zeta Function of Galois covers) Let Y/X be a Galois cover with $G = Aut(Y/X)$ with G abelian. Let $\widehat{G}$ be the set of characters of G. Then*

$$\zeta_Y(u) = \prod_{\chi \in \widehat{G}} L(u, \chi, Y/X).$$

Proof. See Corollary 18.11 on page 155 of [5]. □

The last result of this section is an important property of L-functions we will use in Chapter 3 to prove our spanning tree formula for elementary abelian 2-group covers.

Definition 2.2.12. Let Y/X be a cover with projection map p. A multigraph $\widetilde{X}$ is an intermediate cover of Y/X if there exist two projection maps $p_1 : Y \to \widetilde{X}$ and $p_2 : \widetilde{X} \to X$ such that $p = p_2 \circ p_1$.

See Theorem 14.3 on page 118 of [5] for the Galois correspondence in the context of Galois covers of (connected) multigraphs. Furthermore Artin-Ihara L-functions satisfy the same formalism as Artin L-functions in algebraic number theory. (See Proposition 18.10 on page 154 of [5].) In particular, we will use the inflation property which we now remind the reader about.

Theorem 2.2.13. *Suppose that $\widetilde{X}$ is intermediate to Y/X and assume that $\widetilde{X}/X$ is Galois. Let $G = Aut(Y/X)$ and $H = Aut(Y/\widetilde{X})$. Let $\widetilde{\chi}$ be a character for $G/H \cong Aut(\widetilde{X}/X)$ and let π be the natural projection from G to G/H. If $\chi = \widetilde{\chi} \circ \pi$, then*

$$L(u, \chi, Y/X) = L(u, \widetilde{\chi}, \widetilde{X}/X)$$

Chapter 3

Analytic Spanning Tree Formulas

Throughout this section, we assume that Y/X is a Galois cover with abelian automorphism group G and that $|G| = d$. *From now on, we also assume that X is not a cycle graph.* It then follows from Theorem 1.2.5 that the order of vanishing of $\zeta_X(u)^{-1}$ at $u = 1$ is r_X and the order of vanishing of $\zeta_Y(u)^{-1}$ at $u = 1$ is r_Y. We first discuss the order of vanishing of the L-functions.

Theorem 3.0.1. *The order of vanishing of $L(u, \chi, Y/X)^{-1}$ at $u = 1$ is $r_X - 1$ provided χ is not the trivial character.*

Proof. From Theorem 2.2.11, one has that

$$\zeta_Y(u) = \zeta_X(u) \prod_{\chi \neq \chi_1} L(u, \chi, Y/X).$$

This implies that the order of vanishing of ζ_Y^{-1} at $u = 1$ is given by

$$r_Y = r_X + \sum_{\chi \neq \chi_1} r(\chi)$$

where $r(\chi)$ is the order of vanishing of the inverse the L-function associated to χ. Theorem 2.2.9 implies that $r(\chi) \geq r_X - 1$ for all nontrivial characters χ. Because $|E_Y| = d|E_X|$ and $|V_Y| = d|V_X|$, we can now show that

$$r_Y = r_X + (d - 1)(r_X - 1).$$

If $r(\chi) > r_X + 1$ then one would have

$$r_Y = r_X + \sum_{\chi \neq \chi_1} r(\chi) > r_X + (d-1)(r_X - 1) = r_Y.$$

This is a contradiction and so $r(\chi) = r_X - 1$ for every nontrivial χ. □

Definition 3.0.2. We define

$$L^*(1,\chi) = \frac{d^{r-1}}{du^{r-1}} \frac{L(u,\chi,Y/X)^{-1}}{(r-1)!}\Big|_{u=1},$$

where $r = r_X$. In other words it is the first non-vanishing Taylor coefficient of the inverse of the L-function at $u = 1$.

Similarly $\zeta_Z^*(1)$ will denote the first non-vanishing Taylor coefficient of the inverse of the Ihara Zeta function at $u = 1$ of a multigraph Z.

Lemma 3.0.3.

$$\frac{\zeta_Y^*(1)}{\zeta_X^*(1)} = \frac{d(-2)^{(d-1)(r_X-1)}\kappa_Y}{\kappa_X}.$$

Proof. Use $r_Y - 1 = d(r_X - 1)$ in Theorem 1.2.5. □

Now, we specialize to the case where G is an elementary abelian 2-group.

Theorem 3.0.4. *Assume that $G \cong (\mathbb{Z}/2\mathbb{Z})^n$. Let X_i be the intermediate covers of Y/X ($i = 1, \ldots, 2^n-1$) such that $Aut(X_i/X) \cong Aut(Y/X)/Aut(Y/X_i) \cong \mathbb{Z}/2\mathbb{Z}$. Then*

$$\kappa_Y = \frac{2^{2^n-n-1}}{\kappa_X^{2^n-2}} \prod_{i=1}^{2^n-1} \kappa_{X_i}.$$

Proof. By Theorem 2.2.11 we know that

$$\zeta_Y(u) = \zeta_X(u) \prod_{\chi \neq \chi_1} L(u,\chi,Y/X). \tag{3.1}$$

By Theorem 2.2.13, we can replace all the $L(u,\chi,Y/X)$ with $L(u,\widetilde{\chi}_i,X_i/X)$ for each X_i so Equation (3.1) turns into

$$\zeta_Y(u) = \zeta_X(u) \prod_{i=1}^{2^n-1} L(u,\widetilde{\chi},X_i/X). \tag{3.2}$$

So we need to find $L^*(1, \widetilde{\chi}_i, X_i/X)$, since we know the first non-vanishing Taylor coefficient of the zeta function by Theorem 1.2.5. But we know that $Aut(X_i/X) \cong \mathbb{Z}/2\mathbb{Z}$ so

$$\frac{\zeta^*_{X_i}(1)}{\zeta^*_X(1)} = L^*(1, \widetilde{\chi}_i, X_i/X)$$

by Theorem 2.2.11 applied to X_i/X. Using Lemma 3.0.3, since $|Aut(X_i/X)| = 2$ we then have

$$L^*(1, \widetilde{\chi}_i, X_i/X) = \frac{(-1)^{r_X-1}2^{r_X}\kappa_{X_i}}{\kappa_X}.$$

We can now use Equation (3.2) to obtain

$$\frac{\zeta^*_Y(1)}{\zeta^*_X(1)} = \prod_{i=1}^{2^n-1} \frac{(-1)^{r_X-1}2^{r_X}\kappa_{X_i}}{\kappa_X} \tag{3.3}$$

We can also use Lemma 3.0.3 and the fact that $r_Y - 1 = 2^n(r_X - 1)$ to get

$$\frac{\zeta^*_Y(1)}{\zeta^*_X(1)} = \frac{(-1)^{(2^n-1)(r_X-1)}2^{(2^n-1)(r_X-1)+n}\kappa_Y}{\kappa_X}. \tag{3.4}$$

We know that the right hand side of Equation (3.4) must be equal to the right hand side of Equation (3.3). So

$$\frac{(-1)^{(2^n-1)(r_X-1)}2^{(2^n-1)(r_X-1)+n}\kappa_Y}{\kappa_X} = \prod_{i=1}^{2^n-1} \frac{(-1)^{r_X-1}2^{r_X}\kappa_{X_i}}{\kappa_X} \tag{3.5}$$

Now we can solve for κ_Y in the above equation and we get

$$\begin{aligned}\kappa_Y &= \frac{\kappa_X}{(-1)^{(2^n-1)(r_X-1)}2^{(2^n-1)(r_X-1)+n}} \prod_{i=1}^{2^n-1} \frac{(-1)^{r_X-1}2^{r_X}\kappa_{X_i}}{\kappa_X} \\ &= \frac{2^{2^n-n-1}}{\kappa_X^{2^n-2}} \prod_{i=1}^{2^n-1} \kappa_{X_i}.\end{aligned}$$

This is what we wanted to show. □

This result when $n = 2$ is an analogue of Kuroda's class number formula for number fields, see (1.5) of [3], which gives information about the class numbers of $(\mathbb{Z}/2\mathbb{Z})^2$-extensions of number fields.

Theorem 3.0.4 gives a new way of counting spanning trees of the covering multigraph of a $(\mathbb{Z}/2\mathbb{Z})^n$-cover in terms of the number of spanning trees of the intermediate covers. It would be interesting to investigate the following question: Is there a similar formula to Theorem 3.0.4 when G is any abelian group? This would be analogous to Brauer's class number relation that can be found in [7].

Chapter 4

Code and Examples

In this section we compute three examples with all relevant information including: pictures, Laplacians, spanning tree numbers, zeta functions, special values, covers, and the associated voltage maps and voltage groups. We also compute some intermediate covers.

4.1 Code

Here is some code for sage math ([6]) that will compute the zeta function and will print the special value $\zeta_X^*(1)$ for any graph where the vertices are ordered from $1, 2, 3, \ldots, n$.

```
def zetafunction(G):
    n = G.adjacency_matrix().nrows()
    M = matrix(SR,n,n,var("u"))
    Q = Matrix(n,n)
    for k in G.vertices():
        if k in G.loop_vertices():
            Q[k-1,k-1] = 1
    A = (G.adjacency_matrix()+Q)*M
    D = matrix(n,n)
    for i in G.vertices():
        D[i-1,i-1]=G.degree(i)-1
    K = D*M*u
    T = identity_matrix(n) - A + K
    r = G.size()-G.order()+1
```

```
    F = (T.determinant()*(1-u^2)^(r-1)).factor()
    print ((-1)^(r-1)*(r-1)*2^r*(G.kirchhoff_matrix().adjugate()[1,1])).factor()
    return F
```

We will show how to use this code to compute the zeta function and special value for K_4. If we type the following

```
K = Graph()
K.add_edges([(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)])
zetafunction(K)
```

then sage outputs

```
2^8
(2*u^2 + u + 1)^3*(2*u - 1)*(u + 1)^2*(u - 1)^3
```

Similarly for the multigraph X of Example 4.2.2 below, we have

```
G = Graph(multiedges=True,loops=True)
G.add_edges([(1,2),(1,2),(1,2),(1,1),(2,3),(1,3)])
zetafunction(G)
```

then sage outputs

```
    -1 * 2^4 * 3 * 7
    -(15*u^5 + 9*u^4 + 15*u^3 + 3*u^2 + u - 1)*(u + 1)^3*(u - 1)^4
```

4.2 Examples

Example 4.2.1. As we saw throughout this thesis, the cube graph $Y = C_3$ is a quadratic cover of $X = K_4$, see Figure 4.1 below. As we see in Example 2.1.2, we have our natural projection map from the cube to K_4 and $Aut(Y/X) = \mathbb{Z}/2\mathbb{Z}$. We see that $r_X = 3$, and $r_Y = 5$. Furthermore, the zeta functions of X and Y are

$$\zeta_X(u)^{-1} = \left(2\,u^2 + u + 1\right)^3 (2\,u - 1)(u + 1)^2 (u - 1)^3$$

and

$$\begin{aligned}\zeta_Y(u)^{-1} &= \zeta_X(u)^{-1} L(u, \chi, Y/X)^{-1} \\ &= \left(2\,u^2 + u + 1\right)^3 \left(2\,u^2 - u + 1\right)^3 (2\,u + 1)(2\,u - 1)(u + 1)^5 (u - 1)^5\end{aligned}$$

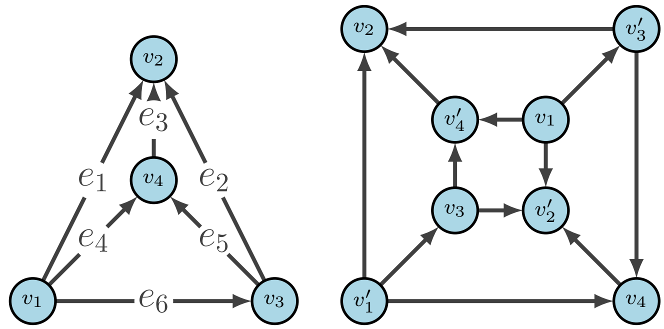

Figure 4.1: The graph C_3 as a $\mathbb{Z}/2\mathbb{Z}$-cover of K_4

The Laplacian matrix of X is

$$L = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

The Laplacian matrix of Y is

$$\begin{bmatrix} 3 & -1 & 0 & -1 & 0 & 0 & 0 & -1 \\ -1 & 3 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 3 & -1 & 0 & -1 & 0 & 0 \\ -1 & 0 & -1 & 3 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 3 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 & -1 & 3 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & 3 & -1 \\ -1 & 0 & 0 & 0 & -1 & 0 & -1 & 3 \end{bmatrix}$$

Using the Laplacian we can compute the spanning tree numbers of X and Y. We can take the adjugate of both of these matrices and find that for X

$$adj(L) = \kappa_X \cdot J = 16 \cdot J.$$

Similarly for Y

$$adj(L) = \kappa_Y \cdot J = 384 \cdot J.$$

We notice that $\kappa_Y/\kappa_X = 24 \in \mathbb{Z}$. Using all of this information we can compute the special values $\zeta_X^*(1)$ and $\zeta_Y^*(1)$ as well as the special value of the L-function at 1 using Theorem 2.2.11 and Lemma 3.0.2. We obtain

$$\zeta_X^*(1) = 2^8.$$

$$\zeta_Y^*(1) = 2^{14} \cdot 3.$$

This implies that

$$L^*(1, \chi, Y/X) = \frac{\zeta_Y^*(1)}{\zeta_X^*(1)} = 2^6 \cdot 3.$$

Example 4.2.2. We now compute a $\mathbb{Z}/3\mathbb{Z}$-cover of the graph seen in Figure 1.2. Call this graph X. Using the voltage assignment $\alpha : \overrightarrow{E} \to \mathbb{Z}/3\mathbb{Z}$ defined by $\alpha(e_3) = \alpha(e_5) = 1, \alpha(e_i) = 0$ for all $i \neq 3, 5$ we get the multigraph shown in Figure 4.2. The number of primes corresponds to the group element in

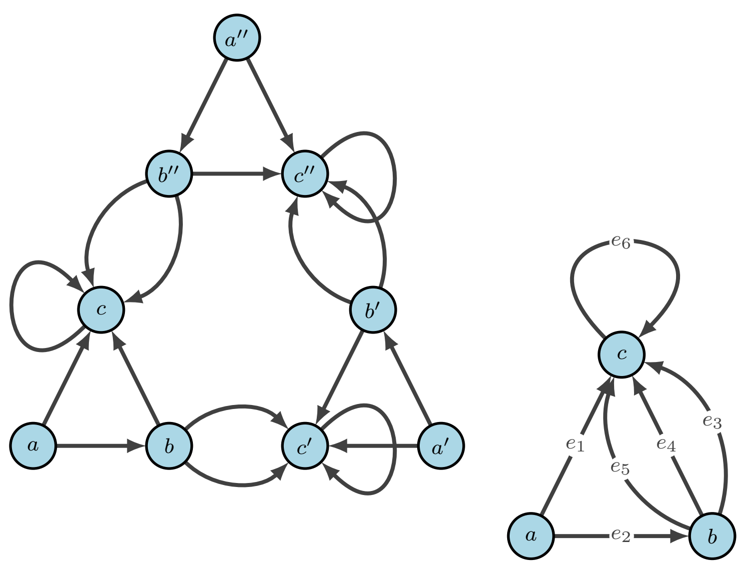

Figure 4.2: The multigraph on the left forms a $\mathbb{Z}/3\mathbb{Z}$-cover of the multigraph on the right.

$\mathbb{Z}/3\mathbb{Z}$. Let Y be the graph covering X. We can compute the zeta function of X and Y as

$$\zeta_X(u)^{-1} = -\left(15\,u^5 + 9\,u^4 + 15\,u^3 + 3\,u^2 + u - 1\right)(u+1)^3(u-1)^4$$

$$\zeta_Y(u)^{-1} = -\big(15\,u^6 - 6\,u^5 + 12\,u^4 - 6\,u^3 + 4\,u^2 - 2\,u + 1\big)^2$$
$$\big(15\,u^5 + 9\,u^4 + 15\,u^3 + 3\,u^2 + u - 1\big)(u+1)^9(u-1)^{10}$$

Note that $L(u, \chi_2, Y/X) = L(u, \chi_3, Y/X)$, where χ_2 and χ_3 are the two non-trivial characters of $\mathbb{Z}/3\mathbb{Z}$.

We can compute the Laplacian matrix of X as follows:

$$L = \begin{bmatrix} 4 & -3 & -1 \\ -3 & 4 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Similarly for Y one gets

$$L = \begin{bmatrix} 4 & -2 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ -2 & 4 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ -1 & 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & -2 & -1 & 0 & -1 & 0 \\ -1 & 0 & -1 & -2 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 4 & -2 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 & -2 & 4 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 2 \end{bmatrix}$$

We compute $\kappa_X = 7$ and $\kappa_Y = 2^2 \cdot 3^3 \cdot 7$. We notice that $\kappa_Y/\kappa_X = 108 \in \mathbb{Z}$. We also see the rank of the fundamental group $r_X = 4$ and $r_Y = 10$. This allows us to compute the special value $\zeta^*_X(1)$ and $\zeta^*_Y(1)$.

$$\zeta^*_X(1) = -2^4 \cdot 3 \cdot 7;\ \ \zeta^*_Y(1) = -2^{12} \cdot 3^5 \cdot 7$$

Example 4.2.3. We now investigate a $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$-cover of the theta graph shown in the figure below and we verify the analogue of Kuroda's class number formula for this particular example. The cube will form a $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$-cover of the theta graph. For this example X will be theta graph and Y will be the cube. We define a voltage $\alpha : \overrightarrow{E_X} \to \mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ as follows:

$$\alpha(e_1) = (\bar{1}, \bar{1});\ \ \alpha(e_2) = (\bar{0}, \bar{0});\ \ \alpha(e_3) = (\bar{1}, \bar{0})$$

Here $(v, (\bar{0}, \bar{0})) = v$; $(v, (\bar{0}, \bar{1})) = v'$; $(v, (\bar{1}, \bar{0})) = v''$; $(v, (\bar{1}, \bar{1})) = v'''$ where v is a or b. One can check that the derived graph obtained is the cube,

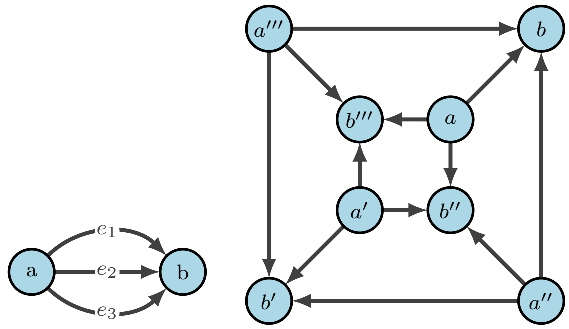

Figure 4.3: The cube C_3 forms a $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$-cover of the theta graph

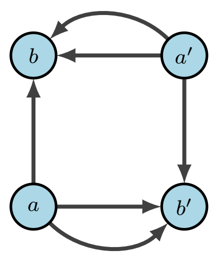

Figure 4.4: The unique intermediate cover to the theta graph.

and that the theta graph has a unique (connected) $\mathbb{Z}/2\mathbb{Z}$-cover which is isomorphic to the graph shown in Figure 4.4. Therefore each of the three proper intermediate covers are isomorphic to this multigraph which has 12 spanning trees and whose Laplacian is

$$\begin{bmatrix} 3 & -2 & 0 & -1 \\ -2 & 3 & -1 & 0 \\ 0 & -1 & 3 & -2 \\ -1 & 0 & -2 & 3 \end{bmatrix}$$

The Laplacian of the theta graph is

$$\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}$$

We observe that $\kappa_Y/\kappa_X = 24 \in \mathbb{Z}$.

We compute

$$\zeta_X(u)^{-1} = -(2\,u+1)(2\,u-1)(u+1)^2(u-1)^2$$

and

$$\zeta_{X_i}(u)^{-1} = \left(2\,u^2+u+1\right)\left(2\,u^2-u+1\right)(2\,u+1)(2\,u-1)(u+1)^3(u-1)^3$$

and $\zeta_Y(u)^{-1}$ has already been computed in Example 4.2.1. We can now use Theorem 3.0.4 to compute the number of spanning trees of the cube in a different way. Let $n = 2$. Then we have from Theorem 3.0.4 that

$$\kappa_Y = \frac{2}{\kappa_X^2} \cdot \prod_{i=1}^{3} \kappa_{X_i} = \frac{2 \cdot 12^3}{3^2} = 384$$

which is the same κ_Y we got from Example 4.2.1.

Appendix A

List of presentations

I gave the following poster presentations and talks on the subject of this honors thesis.

1. State of Jefferson Math Congress (poster), October 2018.
2. MAA Section meeting (poster), AIM, San Jose, February 2019.
3. Chico State Student Research Competition (talk), March 2019.
4. North California Undergraduate Math Conference (talk), Chico, April 2019.
5. Natural Sciences Poster Session (poster), Chico, April 2019.
6. Undergraduate Math Seminar (talk), Chico, May 2019.

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