The Wegner and Minami Estimates in the Anderson Model

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Introduction

The Anderson Model is a physical model in Quantum Mechanics for studying the flow of electrons in impure crystalline structures like salt. In this model, the most famous discovery of Anderson was that as the impurity increases, there is a suppression of electron transport. This phenomenon is called Anderson Localization. For our crystalline structure, we use the underlying lattice $\mathbb{Z}^d$. For studying the energy flow comes down to studying the spectrum (eigenvalues) of random perturbations of the discrete Laplacian on $\ell^2(\mathbb{Z}^d)$. The Wegner Estimate is the key estimate in proving Anderson Localization.

Introduction (Cont'd)

We define

$$H_0(f(k)) = \sum_{|m-k|=1} f(m)$$

that is the sum of the function values of $f \in \ell^2(\mathbb{Z}^d)$ over the nearest neighboring nodes. This operator is a bounded self adjoint operator with spectrum $\sigma(H_0) = [-2d, 2d]$. We now fix a box Λ with finite volume, and look at the restriction of our operator H_0 on Λ and set everything outside of Λ to zero. This operator, denoted as H_0^Λ, becomes self adjoint under this condition.

Introduction (Cont'd)

For the remainder of this talk we will use $n = |\Lambda|$. Fix $\omega = \{\omega_j\}_{j=1}^n$ that are independent identically distributed (iid) and have bounded probability density function ρ, making the probability measure $d\mu_j = \rho(\omega_j)d\omega_j$. Define $V_\omega = \sum_{j=1}^n \omega_j \Pi_j$ where Π_j is the projection onto node j. Then we can look at the new perturbed operator

$$H_\omega^\Lambda = H_0^\Lambda + V_\omega$$

When our dimension is 1 we can write this operator as the matrix

$$H_\omega^\Lambda = \begin{pmatrix} \omega_1 & 1 & & \text{\Large 0} \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ \text{\Large 0} & & 1 & \omega_n \end{pmatrix}$$

Probability Measures

For this problem, we need to set up the notation for our probability measures. We define

$$\mathbb{P}_\Lambda = \prod_{j=1}^{n} d\mu$$

Where a random variable $X : \Omega_\Lambda \to \mathbb{R}$ where Ω_Λ is the Cartesian Product of the probability spaces for μ. Then for any subset $A \subseteq \Omega_\Lambda$ we have

$$\mathbb{P}_\Lambda(A) = \int_{\Omega_\Lambda} \chi_A(\omega)\ d\mu(\omega_1)\dots d\mu(\omega_n)$$

and

$$\mathbb{E}_\Lambda[X] = \int_{\Omega_\Lambda} X(\omega)\ d\mu(\omega_1)\dots d\mu(\omega_n)$$

But for the remainder of these slides we will remove the Λ for brevity.

Posing the Problem

With this notation we can now state our key theorem.

Theorem (Wegner, 1981)

$$\mathbb{P}\{\textit{At least one eigenvalue of } H_\omega^\Lambda \textit{ is in } I\} \leq ||\rho||_\infty |\Lambda||I|$$

To turn this into rigorous mathematical statement, denote $P_\omega^\Lambda(I)$ to be the spectral projection of H_ω^Λ onto the interval I. Having at least one eigenvalue in I is equivalent to $Tr(P_\omega^\Lambda(I)) \geq 1$. Thus we need to show

$$\mathbb{P}\{Tr(P_\omega^\Lambda(I)) \geq 1\} \leq ||\rho||_\infty |\Lambda||I|$$

Chebyschev's Inequality

Instead of working with individual probabilities, we would like to turn our problem into something more tangible to compute. So we will use the following inequality:

Theorem (Chebyschev's Inequality)

If X is a random variable then $\mathbb{P}(X \geq \alpha) \leq \frac{1}{\alpha}\mathbb{E}[X]$

in which $\omega = \{\omega_j\}_{j\in\Lambda}$ Applying this to $X = Tr(P_\omega^\Lambda(I))$ and $\alpha = 1$ we get

$$\mathbb{P}\{Tr(P_\omega^\Lambda(I)) \geq 1\} \leq \mathbb{E}[Tr(P_\omega^\Lambda(I))] \tag{1}$$

Expanding the Trace

We can expand the trace in an orthonormal basis $\{\delta_j\}_{j=1}^{n}$ for $\ell^2(\Lambda)$ as

$$Tr(P_\omega^\Lambda(I)) = \sum_{j=1}^{n} \langle \delta_j, P_\omega^\Lambda(I)\delta_j \rangle$$

and using linearity of expectation, we get

$$\mathbb{E}[Tr(P_\omega^\Lambda(I))] = \mathbb{E}[\sum_{j=1}^{n} \langle \delta_j, P_\omega^\Lambda(I)\delta_j \rangle] = \sum_{j=1}^{n} \mathbb{E}[\langle \delta_j, P_\omega^\Lambda(I)\delta_j \rangle]$$

Fundamental Spectral Averaging Estimate

The following theorem will be the main tool we use to prove this Wegner Estimate.

Theorem

Let ϕ be a vector in a separable Hilbert space $\mathcal{H}$, and consider $H_s = H_0 + s\Pi_\phi$, where H_0 is a self adjoint operator. If μ is any probability measure with compact support that is absolutely continuous with respect to μ_L. Then

$$\mathbb{E}[\langle \phi, P_s(I)\phi \rangle] := \int d\mu(s)\langle \phi, P_s(I)\phi \rangle \leq ||\rho||_\infty |I|$$

Stone's Formula

To be able to write our projector in a way that allows us to compute the integral from the previous theorem we need a result known as **Stone's Formula**.

Theorem (Stone's Formula)

Let H be an operator and let $P([a,b])$ and $P((a,b))$ be spectral projectors for H onto their respective intervals. Then

$$s-\lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_a^b \left(\frac{1}{H - E - i\varepsilon} - \frac{1}{H - E + i\varepsilon} \right) dE = \frac{1}{2} [P([a,b]) + P((a,b))] \tag{2}$$

If a and b are not eigenvalues of H, then we have that

$$P([a,b]) = \frac{1}{2} [P([a,b]) + P((a,b)))]$$

The Rank One Resolvent Identity

The last thing we need is a formula that will come into play later. For simplicity we denote the resolvent as $R_H(z) = \frac{1}{H-z}$, for $z \in \rho(H)$, where $\rho(H)$ is the resolvent set of H.

Lemma

Let H_0 be a self adjoint operator, and $H_s = H_0 + s\Pi_\phi$. Then if we denote R_s and R_0 as the resolvents of H_s and H_0 respectively, we have for $Im(z) \neq 0$

$$\langle \phi, R_s(z)\phi \rangle = \frac{\langle \phi, R_0(z)\phi \rangle}{1 + s\langle \phi, R_0(z)\phi \rangle}$$

Proof of Spectral Averaging

To prove the Wegner Estimate we do the following things

1. For this we will assume that $I = [a, b]$ and neither a or b are eigenvalues of H_s. Now, we use Stone's Formula for the projector in the inner product and rearrange

$$\frac{1}{2\pi i} \lim_{\varepsilon \to 0} \int_{\mathbb{R}} d\mu(s) \int_I dE \langle \phi, \left(\frac{1}{H_s - E - i\varepsilon} - \frac{1}{H_s - E + i\varepsilon} \right) \phi \rangle$$

2. Use the first Resolvent Identity to turn our integrand into

$$\frac{1}{H_s - E - i\varepsilon} - \frac{1}{H_s - E + i\varepsilon} = \frac{2i\varepsilon}{(H_s - E)^2 + \varepsilon^2} = 2i\, Im(\langle \phi, R_s(E + i\varepsilon)\phi \rangle)$$

Proof of Spectral Averaging (Cont'd)

3. Swap the order of integration and use the previous lemma, with $(\langle \phi, R_0(E+i\varepsilon)\phi\rangle)^{-1} = x + iy$ to give

$$\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_I dE \int_{\mathbb{R}} d\mu(s)\ Im\left(\frac{1}{x+iy+s}\right)$$

$$= \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_I dE \int_{\mathbb{R}} d\mu(s)\ \frac{y}{(x+s)^2+y^2}$$

4. Now use $d\mu(s) = \rho(s)ds$ and bound $\rho(s) \le ||\rho||_\infty$ and evaluate the first integral in s to give us that

$$\frac{1}{\pi}\int_{\mathbb{R}} \rho(s)ds\frac{y}{(x+s)^2+y^2} \le \frac{1}{\pi}||\rho||_\infty \cdot \arctan\left(\frac{s+x}{y}\right)\Big|_{-\infty}^{\infty} = ||\rho||_\infty$$

Proof of Spectral Averaging (Cont'd)

5. Substituting this back in we can now deal with the limit as $\varepsilon \to 0$. Since ε was embedded in x and y and those variables were removed by the evaluation of the arctan function, we see that we get

$$\mathbb{E}[\langle \phi, P_s(I)\phi \rangle] \leq ||\rho||_\infty \int_I dE = ||\rho||_\infty |I|$$

.

Proving Wegner Using Spectral Averaging

We now use the spectral averaging estimate and apply it to the following. Writing $H_\omega^\Lambda = H_{\omega_j^\perp} + \omega_j \Pi_j$, we can use $s = \omega_j$, $H_0 = H_{\omega_j^\perp}$, and $\phi = \delta_j$ to give us

$$\mathbb{E}[TrP_\omega^\Lambda(I)] \leq \sum_{j=1}^{n} \mathbb{E}[\langle \delta_j, Tr(P_{\omega_j}^\Lambda(I))\delta_j \rangle] \leq \sum_{j=1}^{n} ||\rho||_\infty |I| = ||\rho||_\infty |\Lambda||I|$$

This proves the Wegner Estimate.

The Minami Estimate

Now we extend our question from one eigenvalue being in an interval to two eigenvalues being in the same interval.

Theorem (Minami, 1996)

$\mathbb{P}\{\textit{At least } \textbf{two} \textit{ eigenvalues of } H_\omega^\Lambda \textit{ are in } I\} \leq (||\rho||_\infty |I||\Lambda|)^2$

Probability Bound

Now we can't immediately apply Chebyschev's inequality to get the quadratic behavior so we need a lemma.

Lemma

Let X be an integer valued random variable. Then $\mathbb{P}(X \geq 2) \leq \mathbb{E}\{X(X-1)\}$

Proof.

We begin with

$$\mathbb{P}(X \geq 2) \leq \sum_{j \geq 2} \mathbb{P}(X \geq j)$$

We can expand this out further and see that

$$\sum_{j \geq 2} \mathbb{P}(X \geq j) = \begin{matrix} \mathbb{P}(X=2)+ & \mathbb{P}(X=3)+ & \mathbb{P}(X=4) & \dots \\ & \mathbb{P}(X=3)+ & \mathbb{P}(X=4) & \dots \\ & & \mathbb{P}(X=4) & \dots \end{matrix}$$

Probability Bound

Proof.

(Cont'd) So we now get

$$\sum_{j\geq 2}\mathbb{P}(X\geq j)=\sum_{j\geq 2}(j-1)\mathbb{P}(X=j)\leq \sum_{j\geq 2}j(j-1)\mathbb{P}(X=j)=\mathbb{E}[X(X-1)]$$

which is what we wanted to show. □

Trace Inequality

The last thing we need is a Lemma devoted to how the Trace of our projector changes as s increases

Lemma

Suppose that $H_s = H_0 + s\Pi_\phi$ *and* $Tr(P_s((-\infty, c])) < \infty$ *for all* $c \in \mathbb{R}$ *and* $s > 0$. *Then for all* $0 \leq s \leq t$,

$$Tr(P_s((a, b])) \leq 1 + Tr(P_t((a, b]))$$

This says that a rank one perturbation can move at most one eigenvalue in or out of an interval.

Using the Trace Inequality

To apply the previous lemma to H^{Λ}_{ω} we need to construct a new random variable that we can replace that is larger than ω_j but still is independent with respect to $\omega_j^{\perp}$. To do this, Let $M = \max\{supp\mu_j\}$. Then take $\rho(\omega_j)d\omega_j$ and send it to $\rho(\tau_j - 3M)d\tau_j$.
So, as an example if ω_j is supported in $[-1, 1]$, then τ_j is supported in $[2, 4]$.

Proving the Minami Estimate

Using $X = Tr(P^{\Lambda}_{\omega_j}(I))$, and τ_j from the previous slide, we get

$$Tr(P^{\Lambda}_{\omega}(I))(Tr(P^{\Lambda}_{\omega}(I)) - 1) = \sum_{j\in\Lambda} \langle \delta_j, P^{\Lambda}_{\omega}(I)\delta_j\rangle (Tr(P^{\Lambda}_{\omega}(I)) - 1)$$

$$\leq \sum_{j\in\Lambda} \langle \delta_j, P^{\Lambda}_{(\omega_j^{\perp},\omega_j)}(I)\delta_j\rangle Tr(P^{\Lambda}_{(\omega_j^{\perp},\tau_j)}(I))$$

So taking the expectation over ω and split the expectation into $\omega_j^{\perp}$ and ω_j to get,

$$\sum_{j\in\Lambda} \mathbb{E}_{\omega_j}\{\langle \delta_j, P^{\Lambda}_{(\omega_j^{\perp},\omega_j)}(I)\delta_j\rangle\}\mathbb{E}_{\omega_j^{\perp}}\{Tr\left(P^{\Lambda}_{(\omega_j^{\perp},\tau_j)}(I)\right)\}$$

Proving the Minami Estimate (Cont'd)

Now we can use the spectral averaging estimate on the first term in the series, to get

$$\sum_{j\in\Lambda} \mathbb{E}_{\omega_j}\{\langle \delta_j, P^{\Lambda}_{(\omega_j^{\perp},\omega_j)}(I)\delta_j\rangle\}\mathbb{E}_{\omega_j^{\perp}}\{Tr\left(P^{\Lambda}_{(\omega_j^{\perp},\tau_j)}(I)\right)\}$$

$$\leq (||\rho||_\infty |I|)\sum_{j\in\Lambda} \mathbb{E}_{\omega_j^{\perp}}\{TrP^{\Lambda}_{(\omega_j^{\perp},\tau_j)}(I)\}$$

And now for the last term we can take the expectation with respect to τ_j since they are shifted iid relative to $\omega_j^{\perp}$, so we get

Proving the Minami Estimate (Cont'd)

$$\mathbb{E}_{\omega}\{X(X-1)\} = \mathbb{E}_{\tau}\{\mathbb{E}_{\omega}(X(X-1))\}$$

which implies along with our previous inequalities that

$$(||\rho||_{\infty}|I|)\sum_{j\in\Lambda}\mathbb{E}_{(\omega_j^{\perp},\tau_j)}\{Tr(P^{\Lambda}_{(\omega_j,\tau_j)}(I))\} \leq$$

$$(||\rho||_{\infty}|I|)(\sum_{j\in\Lambda}||\rho||_{\infty}|I||\Lambda|) = (||\rho||_{\infty}|I||\Lambda|)^2$$

which completes the proof.

Corollary to the Minami Estimate

We can generalize the previous process to get the following corollary.

Corollary

$\mathbb{P}\{\textit{At least } \mathbf{n} \textit{ eigenvalues of } H_\omega^\Lambda \textit{ are in } I\} \leq \frac{1}{n!}(||\rho||_\infty |I||\Lambda|)^n$

To prove this, we need to use the same proof technique that we used in the Minami Estimate in an inductive way. To see this we begin with a nice probability theory result. If X is a discrete random variable, then we have

$$\mathbb{P}(X \geq n) \leq \mathbb{P}(X(X-1)\dots(X-(n-1)) \geq n!) \leq \frac{1}{n!}\mathbb{E}[X(X-1)\dots(X-n-1)].$$

Corollary to the Minami Estimate (Cont'd)

So with this, if we pick τ_j as before where it is iid to $\omega_j^{\perp}$ but with shifted support, we see we can use the following

$$Tr(P_{\omega}^{\Lambda}(I) - k) \leq 1 + Tr(P_{(\omega_j^{\perp}, \tau_j)}^{\Lambda}(I) - k) = Tr(P_{(\omega_j^{\perp}, \tau_j)}^{\Lambda}(I) - (k-1))$$

to shift the each Trace up by 1, apply spectral averaging and apply the inductive step to complete the proof.

Application of These Estimates

1. As we saw in the introduction, the Wegner Estimate is used to prove Anderson Localization. Anderson Localization states that there exists an interval $I \subseteq \mathbb{R}$ so that $\sigma(H_\omega) \cap I$ only contains isolated eigenvalues and the eigenfunctions of these eigenvalues exhibit exponential decay.
2. The Minami Estimate is used in local eigenvalue statistics and can be used to prove the following: Near an eigenvalue in the localization region, there is no correlation between the eigenvalues of H_ω^Λ for large Λ.

THANK YOU!

References

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