A Review of Random Schrödinger Operators in the Anderson Model

Supplementary Material For M.S. in Mathematics and P.h.D Qualifying Exam

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1 Masters Work

1.1 Introduction

We begin with the Anderson Model, which is a model that explains the quantum mechanical effects of impurities in crystalline structures like salt. The most famous phenomenon which arose for this model was Anderson Localization, that is, that the impurities in the structure suppress electron transport. Shortly after this discovery, the mathematical world became increasingly interested in modeling this situation and the Anderson Model we used here was created. In this model, one can state and prove Anderson Localization as a theorem. One of the biggest pieces of this proof is the Wegner Estimate. We use the Lattice $\mathbb{Z}^d$ to represent atoms, and our electron transfer through impurity will be represented by random perturbations of the discrete Laplacian on $\ell^2(\mathbb{Z}^d)$.

To begin our investigations into the Wegner Estimate, we fix a box $\Lambda \subset \mathbb{Z}^d$ so that $|\Lambda|$ is finite. Throughout this section, we will denote $n = |\Lambda|$ (Later we will introduce a function $n(E)$, which will be different). We define H_0 on $\ell^2(\Lambda)$ as

$$H_0 f(k) = \sum_{|m-k|=1} f(m)$$

This operator H_0 is a bounded self-adjoint operator, with its spectrum $\sigma(H_0) = [-2d, 2d]$.

We want to study a perturbation of this operator H_0 by a random potential. Let $\{\omega_j\}_{j=1}^n$ be independent identically distributed (i.i.d.) real random variables with probability measure $d\mu(\omega) = \rho(\omega)d\omega$, where $\rho \in L^\infty(\mathbb{R})$. We define

$$d\mathbb{P}_\Lambda = \prod_{j=1}^n d\mu$$

Where a random variable $X : \Omega_\Lambda \to \mathbb{R}$ where Ω_Λ is the Cartesian Product of the probability spaces for μ. Then for any subset $A \subseteq \Omega_\Lambda$ we have

$$\mathbb{P}_\Lambda(A) = \int_{\Omega_\Lambda} \chi_A(\omega)\ d\mu(\omega_1)\ldots d\mu(\omega_n)$$

and

$$\mathbb{E}_\Lambda[X] = \int_{\Omega_\Lambda} X(\omega)\ d\mu(\omega_1)\ldots d\mu(\omega_n)$$

For the rest of this paper, we will remove the Λ for brevity.

Now, we can define the operator

$$H_\omega^\Lambda = H_0 + V_\omega$$

where

$$V_\omega = \sum_{j=1}^n \omega_j \Pi_j = \sum_{j=1}^n \omega_j \langle \delta_j, \cdot \rangle \delta_j$$

where Π_j is the single site projection onto j. As an example of this we have that in the case where $d = 1$, H_ω^Λ can be written as a matrix as

$$H_\omega^\Lambda = \begin{pmatrix} \omega_1 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & \omega_n \end{pmatrix}$$

For $d > 1$, the matrix representation becomes a banded matrix dependent on the choice of ordering of the vertices in Λ. In all dimensions, ω_j's will only appear on the main diagonal.

1.2 The Wegner Estimate

For this section, $\mathbb{E}$ is the expectation for the ensemble ω. We can now ask the following question: What is the probability that an eigenvalue of H_ω^Λ is in an interval I? Our goal is to eventually prove the following estimate, known as the Wegner Estimate:

Theorem 1.2.1. *[9]* $\mathbb{P}\{$*At least one eigenvalue of* H_ω^Λ *is in* $I\} \leq ||\rho||_\infty |I||\Lambda|$

But first we need to develop the machinery needed to quantify this result mathematically. We denote the spectral projector of H_ω^Λ onto the interval I as $P_\omega^\Lambda(I) = \chi_I(H_\omega^\Lambda)$. With this we can now reduce our problem to calculating the $TrP_\omega^\Lambda(I)$ and this is now a discrete random variable that counts the number of eigenvalues of H_ω^Λ in I. Hence, we can rephrase the statement of Theorem 1.2.1 as

$$\mathbb{P}\{Tr(P_\omega^\Lambda(I)) \geq 1\} \leq ||\rho||_\infty |I||\Lambda|$$

Immediately applying Chebyschev's inequality which states for a random variable X and $\alpha \in \mathbb{R}$, that

$$\mathbb{P}\{X \geq \alpha\} \leq \frac{1}{\alpha}\mathbb{E}[X].$$

We apply this to $X = Tr(P_\omega^\Lambda)$ to get

$$\mathbb{P}\{Tr(P_\omega^\Lambda(I)) \geq 1\} \leq \mathbb{E}[Tr(P_\omega^\Lambda(I))]$$

We can immediately begin analyzing this expectation, by expanding the trace in orthonormal basis $\{\delta_j\}_{j=1}^n$ in $\ell^2(\Lambda)$:

$$TrP_\omega^\Lambda(I) = \sum_{j=1}^n \langle \delta_j, P_\omega^\Lambda(I)\delta_j \rangle$$

So using linearity of expectation we have

$$\mathbb{E}[Tr(P_\omega^\Lambda(I))] = \mathbb{E}[\sum_{j=1}^n \langle \delta_j, P_\omega^\Lambda(I)\delta_j \rangle] = \sum_{j=1}^n \mathbb{E}[\langle \delta_j, P_\omega^\Lambda(I)\delta_j \rangle] \tag{1.1}$$

To bound each term in the series we will need the fundamental spectral averaging estimate. Let $\omega_j^\perp$ be the set of all random variables that are not ω_j. Then we can write

$$H_\omega^\Lambda = H_{\omega_j^\perp} + \omega_j \Pi_j$$

for all $j = 1 \dots n$. Note that $\omega_j \Pi_j$ is a rank-one projection onto the subspace spanned by δ_j. The following result of Combes, Germinet, and Klein can be found in [2].

Theorem 1.2.2. *(Fundamental Spectral Averaging Estimate) Let ϕ be a vector in a separable Hilbert space $\mathcal{H}$, and consider $H_s = H_0 + s\Pi_\phi$ where Π_ϕ is the projector onto the subspace spanned by ϕ. Then*

$$\mathbb{E}[\langle \phi, P_s(I)\phi \rangle] := \int d\mu(s) \langle \phi, P_s(I)\phi \rangle \leq ||\rho||_\infty |I|$$

Before we prove this theorem we will need to state the important result which will allow us to bound the above integral easily, which is known as **Stone's Formula**. More details can be found in [8].

Theorem 1.2.3. *Let H be an operator and let $P([a,b])$ and $P((a,b))$ be spectral projectors for H onto their respective intervals. Then*

$$s - \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_I \left(\frac{1}{H - E - i\varepsilon} - \frac{1}{H - E + i\varepsilon} \right) dE = \frac{1}{2}[P([a,b]) + P((a,b))] \tag{1.2}$$

If a and b are not eigenvalues of H, then we have that

$$P([a,b]) = \frac{1}{2}[P([a,b]) + P((a,b)))]$$

For the remainder of this paper, we will use $R_H(z) = \frac{1}{H-z}$ which is called the **Resolvent** for H. Thus in Theorem 1.2.3, the integrand can be rewritten as $R_H(E+i\varepsilon) - R_H(E-i\varepsilon)$. This notation immediately allows us to simplify writing the calculation in the lemma below.

Lemma 1. *Let H_0 be a self adjoint operator, and $H_s = H_0 + s\Pi_\phi$, with Π_ϕ the same as Theorem 1.2.2. Then if we denote R_s and R_0 as the resolvents of H_s and H_0 respectively, we have*

$$\langle \phi, R_s(z)\phi \rangle = \frac{\langle \phi, R_0(z)\phi \rangle}{1 + s\langle \phi, R_0(z)\phi \rangle}$$

Proof. We begin with our second resolvent identity:

$$R_0(z) - R_s(z) = R_0(z)(H_0 - (H_0 + s\Pi_\phi))R_s(z) = -R_0(z)(s\Pi_\phi)R_s(z)$$

So from this we can now use the fact that Π_ϕ is a rank one projection, and we see that

$$s\Pi_\phi(\cdot) = s\langle \phi, \cdot \rangle \phi$$

So we can now multiply both sides by ϕ and take the inner product of ϕ with both to yield

$$\langle \phi, (R_0(z))\phi \rangle - \langle \phi, (R_s(z))\phi \rangle = -\langle \phi, R_0(z)(s\langle \phi, R_s(z)\phi \rangle)\phi \rangle) = -s\langle \phi, R_0(z)\phi \rangle \langle \phi, R_s(z)\phi \rangle$$

and then you solve for $\langle \phi, R_s(z)\phi \rangle$ to get

$$\langle \phi, R_s(z)\phi \rangle = \frac{\langle \phi, R_0(z)\phi \rangle}{1 + s\langle \phi, R_0(z)\phi \rangle}$$

□

We now have enough to prove Theorem 1.2.2.

Proof. (Theorem 2.2) We choose an interval $I = [a, b]$ so that neither a or b are eigenvalues of $H_s = H_0 + s\Pi_\phi$. Then we begin analyzing

$$\int_{\mathbb{R}} d\mu(s)\langle \phi, P_s(I)\phi\rangle$$

And we plug in Stone's Formula for $P_s(I)$ to get

$$\frac{1}{2\pi i}\lim_{\varepsilon\to 0}\int_{\mathbb{R}} d\mu(s)\int_I dE\langle \phi, \left(\frac{1}{H_s - E - i\varepsilon} - \frac{1}{H_s - E + i\varepsilon}\right)\phi\rangle \tag{1.3}$$

We first combine the two resolvents together to get that

$$\frac{1}{H_s - E - i\varepsilon} - \frac{1}{H_s - E + i\varepsilon} = \frac{2i\varepsilon}{(H_s - E)^2 + \varepsilon^2} = 2i \cdot Im(\langle \phi, R_s(E + i\varepsilon)\phi\rangle)$$

Plugging this into (1.3) we have

$$\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_{\mathbb{R}} d\mu(s)\int_I dE\ Im(\langle \phi, R_s(E + i\varepsilon)\phi\rangle) \tag{1.4}$$

Now we apply Lemma 1 to $\langle \phi, R_s(E + i\varepsilon)\phi\rangle$ to expand our integrand as

$$\langle \phi, R_s(E + i\varepsilon)\phi\rangle = \frac{\langle \phi, R_0(E + i\varepsilon)\phi\rangle}{1 + s\langle \phi, R_0(E + i\varepsilon)\phi\rangle}$$

Multiplying the numerator and denominator by $(\langle \phi, R_0(E + i\varepsilon)\phi\rangle)^{-1} = x + iy$ we see that we get

$$\langle \phi, R_s(E + i\varepsilon)\phi\rangle = \frac{1}{(x + iy) + s}$$

Plugging this into Equation (1.4) and swapping the order of integration we get

$$\frac{1}{\pi}\lim_{\varepsilon\to 0}\int_I dE\int_{\mathbb{R}} d\mu(s)\ Im\left(\frac{1}{x + iy + s}\right) = \frac{1}{\pi}\lim_{\varepsilon\to 0}\int_I dE\int_{\mathbb{R}} d\mu(s)\ \frac{y}{(x + s)^2 + y^2}$$

Looking at only the integral over s and simplifying using the fact that $\rho(s) \in L^\infty(\mathbb{R})$ we get:

$$\int_{\mathbb{R}} \rho(s)ds\frac{y}{(x + s)^2 + y^2} \le ||\rho||_\infty \int_{\mathbb{R}} \frac{y}{(x + s)^2 + y^2} = ||\rho||_\infty \cdot \arctan\left(\frac{s + x}{y}\right)\Bigg|_{-\infty}^{\infty} = ||\rho||_\infty \cdot \pi$$

Therefore we compute the limit as $\varepsilon \to 0$ to get

$$\mathbb{E}[\langle \phi, P_s(I)\phi\rangle] \le ||\rho||_\infty \int_I dE \le ||\rho||_\infty |I|$$

which completes the proof. □

So we can now apply Spectral Averaging term by term in our trace expansion in (1.1). To do this for our setup we have $s = \omega_j$, and $\phi = \delta_j$. We now can also use our measure $d\mu(\omega) = d\mu_\Lambda(\omega) = \prod_{j=1}^n d\mu(\omega_j)$, since our measure is now a product measure over the sites j. Then we can apply the spectral averaging result term by term to give

$$\mathbb{E}[TrP_\omega^\Lambda(I)] \le \sum_{j=1}^{n} \mathbb{E}\{\langle \delta_j, TrP_{\omega_j}^\Lambda(I)\delta_j\rangle\} \le \sum_{j=1}^{n} ||\rho||_\infty |I| = ||\rho||_\infty |\Lambda||I|$$

1.3 The Minami Estimate

We now rephrase our original question but with the natural extension of asking the same question about two eigenvalues in the interval I. More on this section can be found in [3]. The result we prove is known as the Minami Estimate

Theorem 1.3.1. *[7]* $\mathbb{P}\{$*At least* ***two*** *eigenvalues of* H_ω^Λ *are in* $I\} \le (||\rho||_\infty |I||\Lambda|)^2$

This theorem is significant because the probability that two eigenvalues are in the same interval, is bounded by the probability you would get if the eigenvalues were independent. To prove this we use a similar idea to section 1 but we first need a lemma about probabilities.

Lemma 2. *Let X be a discrete random variable. Then* $\mathbb{P}(X \ge 2) \le \mathbb{E}\{X(X-1)\}$

Proof. We begin with

$$\mathbb{P}(X \ge 2) \le \sum_{j\ge 2} \mathbb{P}(X \ge j)$$

We can expand this out further and see that

$$\sum_{j\ge 2} \mathbb{P}(X \ge j) = \begin{array}{cccc} \mathbb{P}(X=2)+ & \mathbb{P}(X=3)+ & \mathbb{P}(X=4) & \dots \\ & \mathbb{P}(X=3)+ & \mathbb{P}(X=4) & \dots \\ & & \mathbb{P}(X=4) & \dots \end{array}$$

In which we see that for each term $P(X=j)$ shows up $j-1$ times, and hence we get

$$\sum_{j\ge 2} \mathbb{P}(X \ge j) = \sum_{j\ge 2} (j-1)\mathbb{P}(X=j) \le \sum_{j\ge 2} j(j-1)\mathbb{P}(X=j) = \mathbb{E}\{X(X-1)\}$$

which is what we wanted to show. □

Hence we can now look at our problem with $X = Tr(P_\omega^\Lambda)$ and see that we need to bound

$$\mathbb{E}\{Tr(P_\omega^\Lambda)(Tr(P_\omega^\Lambda)-1)\} \tag{1.5}$$

We now state two more Lemmas that are needed to apply spectral averaging and Wegner to Equation 1.5

Lemma 3. *Let $H_s = H_0 + sW$, $P_s(J) = \chi_J(H_s)$ and suppose that $TrP_s((-\infty, c]) < \infty$ for all $c \in \mathbb{R}$ and $s > 0$. Then for all $a < b$ and $0 \le s \le t$ we have*

$$TrP_s((a,b]) \le TrP_0((-\infty,b]) - TrP_t((-\infty,b]) + TrP_t((a,b])$$

And as a consequence we have

Lemma 4. *Suppose that $W = \Pi_\phi$ is the rank one projector onto ϕ. Then*

$$TrP_s((a,b]) \le 1 + TrP_t((a,b])$$

Proof. Let $0 \leq s \leq t$ Then for any $c \in \mathbb{R}$ we have

$$0 \leq TrP_s((-\infty, c]) - Tr(P_t(-\infty, c]) \leq 1$$

where the last inequality is the min-max principle for rank one perturbations and then apply the previous Lemma □

Here we will only apply Lemma 4 because our H_ω^Λ is H_0 plus a sum of rank one perturbations. To prove Theorem 1.3.1 we only need one more thing needed to apply Lemma 4. For each fixed j, we can suppose that supp $\mu \subset [0, M]$. Then we let $\omega_j^\perp$ be the set of random variables that are not ω_j. So we let τ_j be identically distributed to ω_j but with $\tau_j \geq M$, so we can apply Lemma 4.

Proof. (Theorem 1.3.1) We begin by giving a bound for our $X(X-1)$ with this new notation. So consider τ_j as above. Then

$$Tr(P_\omega^\Lambda(I))(Tr(P_\omega^\Lambda(I))-1) = \sum_{j\in\Lambda} \langle \delta_j, P_\omega^\Lambda(I)\delta_j\rangle (Tr(P_\omega^\Lambda(I))-1) \leq \sum_{j\in\Lambda} \langle \delta_j, P^\Lambda_{(\omega_j^\perp, \omega_j)} \delta_j\rangle Tr(P^\Lambda_{(\omega_j^\perp, \tau_j)}(I))$$

Where the inequality is the application of Lemma 4. So now we average over ω and use independence of the ω_j's to get

$$\mathbb{E}\{TrP_\omega^\Lambda(I)(TrP_\omega^\Lambda(I) - 1)\} \leq \sum_{j\in\Lambda} \mathbb{E}_{\omega_j^\perp}\{TrP^\Lambda_{(\omega_j^\perp, \tau_j)}\}\mathbb{E}_{\omega_j}\{\langle \delta_j, P_\omega^\Lambda(I)\delta_j\}$$

But applying spectral averaging to that last term we get that

$$\sum_{j\in\Lambda} \mathbb{E}_{\omega_j^\perp}\{TrP^\Lambda_{(\omega_j^\perp, \tau_j)}\mathbb{E}_{\omega_j}(\langle \delta_j, P_\omega^\Lambda(I)\delta_j\rangle)\} \leq ||\rho||_\infty |I| \sum_{j\in\Lambda} \mathbb{E}_{\omega_j^\perp}\{TrP^\Lambda_{(\omega_j^\perp, \tau_j)}(I)\}$$

Now we can use the fact that the set of all τ's is also independent of ω so we get

$$\mathbb{E}_\omega\{X(X-1)\} = \mathbb{E}_\tau\{\mathbb{E}_\omega(X(X-1))\}$$

Therefore we get

$$||\rho||_\infty |I| \sum_{j\in\Lambda} \mathbb{E}_{\omega_j^\perp}\{TrP^\Lambda_{(\omega_j^\perp, \tau_j)}(I)\} = ||\rho||_\infty |I| \sum_{j\in\Lambda} \mathbb{E}_{(\omega_j^\perp, \tau_j)}\{(TrP^\Lambda_{(\omega_j, \tau_j)}(I)\}$$

And so now applying Wegner and summing over j we get exactly that

$$||\rho||_\infty |I| \sum_{j\in\Lambda} \mathbb{E}_{(\omega_j^\perp, \tau_j)}\{(TrP^\Lambda_{(\omega_j, \tau_j)}(I)\} \leq ||\rho||_\infty |I| \sum_{j\in\Lambda} ||\rho||_\infty |I||\Lambda| = (||\rho||_\infty |I||\Lambda|)^2$$

Where the Wegner Estimate is used in the last line.

□

1.3.1 Generalized Minami Estimate

The first two estimates Wegner and Minami

Corollary 1. $\mathbb{P}\{$*At least* $\boldsymbol{n}$ *eigenvalues of* H_ω^Λ *are in* $I\} \leq \frac{1}{n!}(||\rho||_\infty |I||\Lambda|)^n$

Proof. Use the fact that

$$\mathbb{P}(X \geq n) \leq \mathbb{P}(X(X-1)\dots(X-(n-1)) \geq n!) \leq \frac{1}{n!}\mathbb{E}[X(X-1)\dots(X-n-1)]$$

We can now extend our procedure we used to prove the Minami estimate to achieve this result. The base case is the Wegner Estimate, so we just need the inductive step. So we suppose that

$$\mathbb{E}[X(X-1)\dots(X-(n-1))] \leq \frac{1}{n!}(||\rho||_\infty |I||\Lambda|)^n$$

and we will Try to show it for $n+1$. We still use $\tau_j \geq \max \text{supp}\, \mu_j$. Then for all $k = 1, \dots n$,

$$TrP_\omega^\Lambda(I) - k \leq 1 + TrP^\Lambda_{(\omega_j^\perp, \tau_j)}(I) - k = TrP^\Lambda_{(\omega_j^\perp, \tau_j)}(I) - (k-1) \tag{1.6}$$

Note that

$$TrP_\omega^\Lambda(I)(TrP_\omega^\Lambda(I)-1)\dots(TrP_\omega^\Lambda(I)-(n-1)) \geq 0$$

since it is zero unless we have at least n eigenvalues in I. So we have

$$\begin{aligned} &TrP_\omega^\Lambda(I)(TrP_\omega^\Lambda(I)-1)\dots(TrP_\omega^\Lambda(I)-(n-1)) \\ &\leq \sum_j \langle \delta_j, P_\omega^\Lambda(I)\delta_j\rangle (TrP^\Lambda_{(\omega_j^\perp, \tau_j)}(I))(TrP^\Lambda_{(\omega_j^\perp, \tau_j)}(I)-1)\dots \\ &\times (TrP^\Lambda_{(\omega_j^\perp, \tau_j)}(I)-(n-1)) \end{aligned}$$

We can now apply Theorem 1.2.2 to get

$$\begin{aligned} &\mathbb{E}\{\langle \delta_j, P_\omega^\Lambda(I)\delta_j\rangle (TrP^\Lambda_{(\omega_j^\perp, \tau_j)}(I))(TrP^\Lambda_{(\omega_j^\perp, \tau_j)}(I)-1)\dots(TrP^\Lambda_{(\omega_j^\perp, \tau_j)}(I)-(n-1))\} \\ &\leq ||\rho||_\infty |I| \mathbb{E}_{\omega_j^\perp}\{(TrP^\Lambda_{(\omega_j^\perp, \tau_j)}(I))(TrP^\Lambda_{(\omega_j^\perp, \tau_j)}(I)-1)\dots(TrP^\Lambda_{(\omega_j^\perp, \tau_j)}(I)-(n-1))\} \end{aligned}$$

So now take $\tau = \{\boldsymbol{\tau_j + \tilde{\omega}_j}\}_{\mathbf{j \in \Lambda}}$ where $\{\tilde{\omega}_j\}_{j\in\Lambda}$ are independent random variables, independent of ω, so that $\tilde{\omega}_j$ has μ_j for probability distribution, and $a_j = \max \text{supp}\, \mu_j$. Using our previous results along with the inductive hypothesis we get

$$\begin{aligned} &\mathbb{E}_\omega\{TrP_\omega^\Lambda(I)(TrP_\omega^\Lambda(I)-1)\dots(TrP_\omega^\Lambda(I)-(n-1))(TrP_\omega^\Lambda(I)-n)\} \\ &= \mathbb{E}_{(\omega,\tau)}\{TrP_\omega^\Lambda(I)(TrP_\omega^\Lambda(I)-1)\dots(TrP_\omega^\Lambda(I)-(n-1))(TrP_\omega^\Lambda(I)-n)\} \\ &\leq ||\rho||_\infty |I| \sum_j \mathbb{E}_{(\omega_j^\perp, \tau_j)}\{(TrP^\Lambda_{(\omega_j^\perp, \tau_j)}(I))(TrP^\Lambda_{(\omega_j^\perp, \tau_j)}(I)-1)\dots(TrP^\Lambda_{(\omega_j^\perp, \tau_j)}(I)-(n-1))\} \\ &\leq ||\rho||_\infty |I| \sum_{j=1}^{|\Lambda|}(||\rho||_\infty |I||\Lambda|)^n = (||\rho||_\infty |I||\Lambda|)^{n+1} \end{aligned}$$

□

2 Qualifying Work

2.1 Anderson Localization

In this section we give a brief discussion about an important theorem used in Minami's Paper: Localization. The question we wish to answer is quite natural. As $\Lambda \nearrow \mathbb{Z}^d$, in general we no longer have just eigenvalues as part of our spectrum. Therefore, our spectral measure could have an absolutely continuous or singular continuous part. Localization says that in certain parts of our spectrum for H_ω, this does not occur. But first we must define the almost sure spectrum.

Theorem 2.1.1. *There is a closed set $\Sigma \subset \mathbb{R}$ so that $\sigma(H_\omega) = \Sigma$ for $\mathbb{P}$ almost all ω.*

We also remind the reader briefly about the general decomposition of measures.

Theorem 2.1.2. *For any Borel Measure μ, one has that*

$$\mu = \mu_{ac} + \mu_{sc} + \mu_{pp}$$

where μ_{ac} is absolutely continuous with respect to the Lebesgue measure, μ_{sc} is singular with respect to Lebesgue, but is continuous, and μ_{pp} is singular with respect to Lebesgue but is supported on discrete set of Lebesgue measure zero.

So with this we can state the key parts of Localization

Theorem 2.1.3. *(Localization) Let Σ be the almost sure spectrum from the above theorem. Let $\mu(H_\omega)$ be the spectral measure for H_ω and let $\mu_{pp}(H_\omega)$ be the pure point part of $\mu(H_\omega)$. Then,*

- *Near the "edge" of Σ, there exists an interval $[a, b]$ so that*

  $$\mu(H_\omega) \cap [a, b] = \mu_{pp}(H_\omega)$$

  with probability 1. That is the spectral measure of H_ω is exclusively pure point.

- *(Aizenman, Molchanov) [1] There are $s \in (0, 1)$, $C > 0$, $m > 0$ and $r > 0$ such that*

  $$\mathbb{E}[|G^\Lambda(z; x, y)|^s] \leq Ce^{-m|x-y|}$$

  for any hypercube $\Lambda \subset \mathbb{Z}^d$, $x \in \Lambda$ and $y \in \partial\Lambda$ and

  $$z \in \{z| \; Im(z) > 0, |z - E| < r\}$$

  *Where the $G^\Lambda(z; x, y) = \langle \delta_x, R_{H_\omega^\Lambda}(z)\delta_y \rangle$ which is called the **Green's Function** of our equation and is defined to be the x, y matrix element of the Resolvent.*

2.2 The Geometric Resolvent Equation

We begin with a discussion on the main tool of exploiting the properties of localization used in [7]. To do this we need what is called the Geometric Resolvent Equation. The primary goal of this equation is to be able to write the diagonal matrix elements of the Resolvent on a large scale in terms of the diagonal matrix elements of the Resolvent at a small scale plus some controllable boundary term that we will be able to apply the Aizenman and Molchanov version of Theorem 2.1.3. It uses the Green's function discussed in the previous section.

Theorem 2.2.1. *Let $x \in \Lambda_p \subset \Lambda_L$, and let $(y, y') \in \partial\Lambda_p$ mean that $y \in \Lambda_p$, $y' \in \Lambda_L \setminus \Lambda_p$ and y,y' are nearest neighbors, then*

$$G^{\Lambda_L}(z; x, x) = G^{\Lambda_p}(z; x, x) + \sum_{(y,y') \in \partial\Lambda_p} G^{\Lambda_p}(z; x, y) G^{\Lambda_L}(z; y', x)$$

Proof. For this proof we will simplify to let $\chi_p = \chi_{\Lambda_p}$ and $R_p(z) = R_{H_\omega^{\Lambda_p}}(z)$ and $R_L(z) = R_{H_\omega^{\Lambda_L}}(z)$

$$(H_{\Lambda_p} - z)\chi_p = \chi_p(H_{\Lambda_L} - z) + [\chi_p, H_0]$$

Now we multiply on the right by $R_L(z)$ and on the left by $R_p(z)$ which turns this into

$$\chi_p R_L(z) = R_p(z)\chi_p + R_p(z)[\chi_p, H_0]R_L(z)$$

So now we can take each of these terms by δ_x and use the fact that $x \in \Lambda_p$ to get

$$R_L(z)\delta_x = R_p(z)\delta_x + R_p(z)[\chi_p, H_0]R_L(z)\delta_x$$

so taking each of these terms into $\langle \delta_x, \cdot \rangle$ we get

$$G^{\Lambda_L}(z; x, x) = G^{\Lambda_p}(z; x, x) + \langle \delta_x, R_p(z)[\chi_\Lambda, H_0]R_{\Lambda_L}(z)\delta_x \rangle$$

And now we only have to worry about the last term. So now we can insert the identity operator and expand in a basis to get

$$\sum_{y' \in \Lambda_L} \sum_{y \in \Lambda_p} G^{\Lambda_p}(z; x, y)\langle \delta_y, [\chi_p, H_0]\delta_{y'} \rangle G^{\Lambda_L}(z; y', x)$$

So to compute that middle term in the sum we get a few different cases:

- y, y' are not nearest neighbors with $y \in \Lambda_L \setminus \Lambda_p$.
- y, y' are nearest neighbors with $y' \in \Lambda_L \setminus \Lambda_p$
- y, y' are nearest neighbors with both in Λ_p
- y, y' are not nearest neighbors with both in Λ_p

we compute these cases separately. It is easy to see that if y, y' are not nearest neighbors then

$$\langle \delta_y, [\chi_p, H_0]\delta_{y'} \rangle = 0$$

by properties of our Laplacian. So we only have 2 cases left. We look at if both are nearest neighbors in Λ_p Then we see that $\langle \delta_y, [\chi_p, H_0]\delta_{y'} \rangle = \langle \delta_y, H_0\delta_{y'} \rangle - \langle \delta_y, \chi_p H_0 \delta_{y'} \rangle$ but since $y' \in \Lambda_p$ we have that

$$\langle \delta_y, H_0\chi_p\delta_{y'} \rangle = \langle \delta_y, H_0\delta_{y'} \rangle = \langle \delta_y, \chi_p H_0 \delta_{y'} \rangle$$

since $y \in \Lambda_p$ we don't need to worry about the nearest neighbors of y outside of Λ_p. This leaves only the nearest neighbors of y that are nearest neighbors to y' with $y' \in \Lambda_L \setminus \Lambda_p$ in which we get 1 as the result since

$$\langle \delta_y, \chi_p H_0 \delta_{y'} \rangle - \langle \delta_y, H_0 \chi_p \delta_{y'} \rangle = \langle \delta_y, \delta_y \rangle - 0 = 1$$

This proves the identity since we sum over precisely these terms in our cases and therefore we get

$$G^{\Lambda_L}(z; x, x) = G^{\Lambda_p}(z; x, x) + \sum_{(y,y') \in \partial \Lambda_p} G^{\Lambda_p}(z; x, y) G^{\Lambda_L}(z; y', x)$$

□

There are many versions of Theorem 2.2.1. We provide a different version that uses the exact same techniques with H_{Λ_p} and H_0.

Corollary 2. *Suppose that $\Lambda_L = \mathbb{Z}^d$ and $x \in \Lambda_p$. Then we have*

$$G^{\Lambda_p}(z; x, x) = G(z; x, x) - \sum_{(y,y') \in \partial \Lambda_p} G^{\Lambda_p}(z; x, y) G(z; y', x)$$

Both Theorem 2.2.1 and Corollary 2 will allow us to utilize our theorem on localization stated above.

2.3 Ergodic Stochastic Processes

We need a theorem for stochastic processes to have the canonical definition of the density of states. We give a few basic definitions and state the key theorem.

Definition 1. Suppose that X_i are a family of random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then, if $T_j : \Omega \to \Omega$ is measure preserving, i.e. $\mathbb{P}(T_j^{-1} A) = \mathbb{P}(A)$ for all $A \in \mathcal{F}$. These T_j are called ergodic if any invariant $A \in \mathcal{F}$ has probability 0 or 1. We call such a collection of random variables and maps an **ergodic process**.

Hence we can now state the Birkhoff Ergodic Theorem.

Theorem 2.3.1. *(Birkhoff Ergodic Theorem) If $\{X_i\}_{i \in \mathbb{Z}^d}$ is an ergodic process with $\mathbb{E}(X_0) < \infty$ then*

$$\lim_{L \to \infty} \frac{1}{(2L+1)^d} \sum_{i \in \Lambda_L} X_i \to \mathbb{E}(X_0)$$

for $\mathbb{P}$-almost all ω.

2.4 The Density of States

These next sections are based on the work of Kirsch. See [6] for more details.
The density of states measure and its associated functions are incredibly important objects in the study of random Schrodinger operators. We wish to give a self contained background on this object and its properties. Let Λ be a cube in $\mathbb{Z}^d$ Let ϕ be in $C_0(\mathbb{R})$. Define the quantity

$$\nu_L(\phi) = \frac{1}{|\Lambda|} Tr(\chi_\Lambda \phi(H_\omega) \chi_\Lambda) = \frac{1}{|\Lambda|} Tr(\phi(H_\omega) \chi_{\Lambda_L})$$

Where $\phi(H_\omega)$ is defined via the Weierstrass approximation theorem. Our goal is to use the Reisz-Markov Representation theorem for this linear functional which states that

Theorem 2.4.1. *(Riesz-Markov) Let ψ be a positive linear functional ($f \geq 0$ implies that $\psi(f) \geq 0$) on $C_0(\mathbb{R})$. Then there exists a unique radon measure so that*

$$\psi(f) = \int_{\mathbb{R}} f(x) d\mu(x)$$

To check that we can apply this theorem we only need to check that if $\phi \geq 0$, then:

$$Tr(\phi(H_\omega)\chi_{\Lambda_L}) = \sum_{i \in \Lambda_L} \langle \delta_i, \phi(H_\omega)\delta_i \rangle$$

and since we have that $\phi(H_\omega)$ is the norm limit approximation by polynomials, we see that $H_\omega \delta_i = V_\omega \delta_i = \omega_i$ which gives that $\phi(H_\omega) = \phi(\omega_i) \geq 0$ since ϕ is positive. Hence the sum

$$\sum_{i \in \Lambda_L} \langle \delta_i, \phi(H_\omega)\delta_i \rangle = \sum_i \phi(\omega_i) \langle \delta_i, \delta_i \rangle \geq 0$$

so we can apply the Riesz-Markov Theorem. This gives us a sequence of measures $d\nu_L(\lambda)$. We will show that this measure converges vaguely to a measure ν as $L \to \infty$. We now remind the reader of vague convergence.

Definition 2. A sequence ν_n of Borel Measures converges vaguely to a Borel Measure ν if

$$\int \phi(x) d\nu_n(x) \to \int \phi(x) d\nu(x)$$

for all $\phi \in C_0(\mathbb{R})$.

We now state the proposition that gives us the almost sure convergence we wish for.

Proposition 1. *If ϕ is a bounded measurable function, then for $\mathbb{P}$ almost every ω*

$$\lim_{L \to \infty} \frac{1}{|\Lambda_L|} Tr(\phi(H_\omega)\chi_{\Lambda_L}) = \mathbb{E}(\langle \delta_0, \phi(H_\omega)\delta_0 \rangle)$$

Proof. We see that

$$\frac{1}{|\Lambda|} Tr(\phi(H_\omega)\chi_{\Lambda_L}) = \frac{1}{(2L+1)^d} \sum_i \langle \delta_i, \phi(H_\omega)\delta_i \rangle$$

It suffices to show that $X_i = \langle \delta_i, \phi(H_\omega)\delta_i \rangle$ forms an ergodic stochastic process with T_j being the shift operators(From Section 2.3). With this we see we get

$$\begin{aligned} X_i(T_j\omega) &= \langle \delta_i, \phi(H_\omega)\delta_i \rangle \\ &= \langle \delta_i, U_j \phi(H_\omega) U_j^* \delta_i \rangle \\ &= \langle U_j^* \delta_i, \phi(H_\omega) U_j^* \delta_i \rangle \\ &= \langle \delta_{i-j}, \phi(H_\omega)\delta_{i-j} \rangle \\ &= X_{i-j}(\omega) \end{aligned} \tag{2.1}$$

Where we chose U so that $U_j^* \delta_i(n) = \delta_i(n+j) = \delta_{i-j}(n)$ Hence we can apply theorem

2.3.1. Thus we see that

$$\frac{1}{|\Lambda_L|}Tr(\phi(H_\omega)\chi_{\Lambda_L}) = \frac{1}{(2L+1)^d}\sum_{i\in\Lambda_L} X_i \to \mathbb{E}(X_0) = \mathbb{E}(\langle\delta_0, \phi(H_\omega)\delta_0\rangle)$$

□

So we have proven that this holds for fixed ϕ on a set of full probability. Call this set Ω_ϕ. This will depend on ϕ. We can conclude that this holds for all $\phi \in \bigcap_\phi \Omega_\phi$. But this is an uncountable intersection so we need to be careful, since this intersection of full measure sets could have not full measure and we don't even know if this set is measurable. We now turn our attention to investigating the properties of this set, and proving that it holds for almost every ϕ with probability 1.

Theorem 2.4.2. *The measures ν_L converge vaguely to the measure ν $\mathbb{P}$ almost surely, i.e. there is a set Ω_0 of probability one, such that*

$$\int \phi(\lambda)d\nu_L(\lambda) \to \int \phi(\lambda)d\nu(\lambda)$$

for all $\phi \in C_0(\mathbb{R})$ and all $\omega \in \Omega_0$.

Remark 1. *The measure ν is non random.*

Proof. Take a countable dense set $D_0 \subset C_0(\mathbb{R})$ in the sup-norm topology. We know before by Proposition 1 each $\phi \in D_0$ has a set Ω_ϕ with full probability measure so that we have vague convergence. Now take

$$\Omega_0 = \bigcap_{\phi\in D_0} \Omega_\phi.$$

Since Ω_0 is a countable intersection of sets of full measure, Ω_0 has probability one. So for $\omega \in \Omega_0$ the convergence holds for all $\phi \in D_0$. Therefore if $\phi \in C_0(\mathbb{R})$, then there exists a sequence $\phi_n \in D_0$ so that $\phi_n \to \phi$ uniformly. Therefore we get

$$\begin{aligned}
&|\int \phi(\lambda)d\nu(\lambda) - \int \phi(\lambda)d\nu_L(\lambda)| \\
\leq\ &|\int \phi(\lambda)d\nu(\lambda) - \int \phi_n(\lambda)d\nu(\lambda)| \\
+\ &|\int \phi_n(\lambda)d\nu(\lambda) - \int \phi_n(\lambda)d\nu_L(\lambda)| \\
+\ &|\int \phi_n(\lambda)d\nu_L(\lambda) - \int \phi(\Lambda)d\nu_L(\lambda)| \\
&\leq ||\phi-\phi_n||_\infty + ||\phi-\phi_n||_\infty \\
+\ &|\int \phi_n(\lambda)d\nu(\lambda) - \int \phi_n(\lambda)d\nu_L(\lambda)|
\end{aligned} \tag{2.2}$$

So we can make the first two terms small by taking n large enough. And we can make the third term small by making L large enough. □

Now we know that we can define a few related quantities for this measure

Definition 3. The measure ν defined by

$$\nu(A) = \mathbb{E}(\langle\delta_0, \chi_A(H_\omega)\delta_0)$$

for A a Borel set in $\mathbb{R}$ is called the **density of states measure**.
The distribtution function N of ν defined by

$$N(E) = \nu((-\infty, E])$$

is called the **Integrated Density of States**.

We now can see that we can use Theorem 2.4.2 to get a limit definition for $N(E)$.

Corollary 3. *For $\mathbb{P}$-almost every ω the following is true:*
For all $E \in \mathbb{R}$

$$N(E) = \lim_{L \to \infty} \nu_L((-\infty, E]). \tag{2.3}$$

Remark 2. *Note that for fixed E the convergence holds for almost every ω. But this corollary tells us something stronger: It claims the existence of an $E-$independent set of ω such that this limit (2.3) holds for all E.*

Proof. (of Corollary) We prove this limit for values of E so that N is continuous at E. Since N is monotone increasing, the set of discontinuity points of N is at most countable (This is a fact commonly proved in Real Analysis so we omit the proof). Consequently, there is a countable set S of continuity points of N is dense. Therefore there is a set of full probability measure such that

$$\int \chi_{(-\infty,E]}(\lambda) d\nu_L(\lambda) \to N(E)$$

for all $E \in S$.
take $\varepsilon > 0$. Suppose E is any continuity point of N. Then, we find $E_+, E_- \in S$ with $E_- \leq E \leq E_+$ such that $N(E_-) - N(E_+) < \frac{\varepsilon}{2}$. So now we can use the monotonicity of N to see that

$$\begin{gathered} N(E) - \int \chi_{(-\infty,E]}(\lambda) d\nu_L(\lambda) \\ \leq N(E_+) - \int \chi_{(-\infty,E]}(\lambda) d\nu_L(\lambda) \\ \leq N(E_+) - N(E_-) + |N(E_-) - \int \chi_{(-\infty,E]}(\lambda) d\nu_L(\lambda)| \\ \leq \varepsilon \end{gathered}$$

for large L. Similarly, one can extend this argument to see that

$$N(E) - \int \chi_{(-\infty,E]}(\lambda) d\nu_L(\lambda) \geq -\varepsilon$$

and so we see that

$$|N(E) - \int \chi_{(-\infty,E]}(\lambda) d\nu_L(\lambda)| \to 0$$

as $L \to \infty$. This proves the result for the continuity points, but one can easily extend this to all points by Proposition 1.

□

We now turn our attention to a discussion of using the Wegner Estimate, Theorem 1.2.1, to easily show the regularity of the density of states, and the existence of its radon-nikodym derivative, a function we will use extensively later.

Theorem 2.4.3. *The integrated density of states is absolutely continuous if* $||\rho||_\infty < \infty$.

Proof. Let $\varepsilon > 0$, then we see that

$$N(E+\varepsilon)-N(E-\varepsilon) = \int_{(-\infty,E+\varepsilon]} d\nu(\lambda) - \int_{(-\infty,E-\varepsilon]} d\nu(\lambda) = \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|}\mathbb{E}(Tr(P_\omega^\Lambda([E-\varepsilon, E+\varepsilon])))$$

$$\leq \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|}(2||\rho||_\infty |\Lambda|\varepsilon) \leq 2||\rho||_\infty \varepsilon$$

So this proves the absolute continuity of N. □

Since N is absolutely continuous, we can see that $N(E) = \int_{-\infty}^{E} n(\lambda)d\lambda$ for some function $n(\lambda)$. We call this function, $n(E)$ the **density of states function**. This will be an important quantity later, as it will end up being the density of our Poisson point process.

2.5 Point Processes

The last thing we need is a discussion of Point Processes, and an important theorem regarding when a family of point processes is a Poisson point process, allowing us to utilize a nice decomposition of our box Λ_L into independent smaller boxes Λ_p. This section is based on [4] and [5]. We begin with defining point processes on $\mathbb{R}$ (or any complete seperable metric space).

Definition 4. Let X_i be a discrete family of random variables defined on $\mathbb{R}$. Then we define a point process to be

$$\xi = \sum_{i=1}^{n} \delta_{X_i}$$

This also is a random measure, but in this case it describes n random objects in $\mathbb{R}$. For us, these will represent rescaled eigenvalues of H_ω^Λ. One might ask what happens when each of these X_i are i.i.d. as we have been assuming throughout this paper. This leads us to the Poisson Point Process.

Definition 5. Define $K(A)$ to be the number of events occurring in A. The Poisson point process can be defined by assuming there is a boundedly finite Borel Measure μ such that for every finite family of disjoint bounded borel sets $\{A_i, i = 1, \dots k\}$

$$\mathbb{P}\{K(A_i) = n_i, i = 1\dots k\} = \prod_{i=1}^{k} \frac{[\mu(A_i)]^{n_i}}{n_i!} e^{-\mu(A_i)}$$

That is, the number of events in disjoint sets forms a Poisson random variable. We call μ the intensity measure of the process.

There is an extensive theory one can look into regarding these but we will only need a few tools involving weak convergence of point processes and when a point process converges weakly to the Poisson point process.

Definition 6. (Weak Convergence of Point Processes) Let ξ_n be a sequence of point processes defined on $(\Omega, \mathcal{F}, \mathbb{P})$. This sequence converges weakly to ξ defined on a possibly different probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ if for any bounded continuous function ϕ

$$\lim_{n\to\infty} \int \phi(\xi_n)\mathbb{P}(d\omega) = \int \phi(\xi_n)\hat{\mathbb{P}}(d\hat{\omega})$$

This is also known to be equivalent to the following two statements:

1. For any $\phi \in C_0^+(\mathbb{R})$,
$$\lim_{n\to\infty} \mathbb{E}_{\mathbb{P}}[e^{-\xi_n(\phi,\omega)}] = \mathbb{E}_{\hat{\mathbb{P}}}[e^{-\xi(\phi,\hat{\omega})}]$$
where
$$\xi_n(\phi) = \int \phi(x)\xi_n(dx)$$

2. For any $l \geq 1, n_j \geq 0$, and disjoint intervals $I_j, j = 1, \ldots, l$ such that
$$\hat{\mathbb{P}}(\xi(\partial I_j) > 0) = 0$$
one has
$$\lim_{n\to\infty} \mathbb{P}(\xi_n(I_j) = n_j, j = 1, \ldots, l) = \hat{\mathbb{P}}(\xi(I_j) = n_j, j = 1, \ldots, l)$$

Definition 7. (See Section 11.2 of [5]) A family of point processes $\xi_n = \sum_{i=1}^{m_n} \xi_{n_i}$ is called a uniformly asymptotically negligible array if

$$\lim_{n\to\infty} \sup_i \mathbb{P}(K_{n_i}(A) > 0) = 0$$

Theorem 2.5.1. *(Theorem 11.2.V in [5]) A uniformly asymptotically negligible array converges weakly to the poisson point process with intensity measure μ if and only if for all bounded borel sets A with $\mu(\partial A) = 0$,*

$$\sum_{i=1}^{m_n} \mathbb{P}(K_{n_i}(A) \geq 2) \to 0 \quad (n \to \infty).$$

and

$$\sum_{i=1}^{m_n} \mathbb{P}(K_{n_i}(A) \geq 1) \to \mu(A) \quad (n \to \infty).$$

We omit the proof due to technicality, but the important intuition to have about this that will be rigorously proven later is that the first condition says that there are no doubled points, that is there will be no clusters of points. Similarly the second condition says that the intensity measure is characterized by the probability that the event happens once. We now have all the material needed to prove the main theorem of Minami in [7].

2.6 Proof of Minami's Result

We can now define our point process that we will be studying for the remainder of the next two sections. Define

$$\xi(\Lambda, E)(dx) = \sum_j \delta_{|\Lambda|(E_j(\Lambda - E))}(x)dx$$

Where $E_j(\Lambda)$ are the eigenvalues of H_ω^Λ and E is a fixed number. Now we state the main theorem we will prove in detail in the next section.

Theorem 2.6.1. *Suppose that the density of states function $n(E)$ exists at E and is positive, and that localization holds at E in the sense of Theorem 2.1.3. Then, the point* *process $\xi(\Lambda, E)$ converges weakly to the Poisson point process ξ with intensity measure $n(E)dx$.*

Remark 3. *We note that indeed such an E exists as in [1], one can find two conditions for which such an E exists.*

- *if $\rho(x)$ is bounded and if there is a compact interval $[a, b]$ such that $\rho(x)$ is non-decreasing on $(-\infty, a]$ and is non-increasing on $[b, \infty)$, then there is an $E(\rho)$ such that localization holds for $|E| > E(\rho)$.*
- *if $||\rho||_\infty$ is sufficiently small, then localization holds for all E.*

With this, we now describe the idea of this theorem. Since we know we are in the localization region, the spectrum will be pure point near our energy E, and the corresponding eigenfunctions exhibit exponential decay. We now divide our box $[0, \Lambda]^d$ into smaller boxes Λ_p of side length $\sim L^\alpha$, $0 < \alpha < 1$, then most eigenfunctions ψ_{E_j} will be centered in one of the Λ_p away from the boundary so that on $\partial\Lambda_p$, $|\psi_{E_j}|$ will be very small. Thus, the error in working with $\bigoplus H^{\Lambda_p}$ is negligible in comparison to working with H^{Λ_L}. And therefore if we set

$$\eta(p, E)(dx) = \sum_j \delta_{|\Lambda_L|(E_j(\Lambda_p)-E)}(dx)$$

then $\xi(\Lambda, E)$ can be approximated by

$$\eta(L, E) = \sum_p \eta(p, E)(dx)$$

Notice two things here, The scaling of $\eta(p, E)$ still involve the box at large scale, and that $\eta(p, E)$ are independent for different p. We first turn our attention to the asymptotic negligibly of $\eta(p, E)$. To see this we compute for any bounded Borel set A

$$\mathbb{P}(\eta(p, E)(A) \geq 1) = \mathbb{P}(Tr(P^{H_{\Lambda_p}}(\frac{A}{|\Lambda_L|} + E)) \geq 1)$$

$$\leq \mathbb{E}(Tr(P^{H_{\Lambda_p}}(\frac{A}{|\Lambda_L|} + E))) \leq \frac{|A||\Lambda_p|}{|\Lambda_L|}||\rho||_\infty \to 0$$

as $L \to \infty$. We will use this same strategy to get a more explicit bound later, but for now, this confirms that $\eta(p, E)$ is a uniformly asymptotically negligible array and that we now have to verify theorem 2.5.1.

Step 1. We define the class of test functions $\mathcal{A}$ to be functions of the form

$$f(x) = \sum_{j=1}^n \frac{a_j \tau}{(x - \sigma_j)^2 + \tau^2}$$

with $n \geq 1$, $\tau > 0$, $a_j > 0$ and $\sigma_j \in \mathbb{R}$ for $j = 1, \ldots, n$. We now state a theorem regarding point processes and weak convergence with respect to this space of test functions.

Lemma 5. *Let ξ_n and ξ be point processes on $\mathbb{R}$ with intensity measures $\mu_n(dx)$ and $\mu(dx)$ respectively. Suppose that μ_n and μ are Lipschitz continuous with constant c. Then the following are equivalent*

1. *ξ_n converges weakly to ξ*

2. for any $f \in \mathcal{A}$, one has

$$\lim_{n\to\infty} \mathbb{E}[exp(-\xi_n(f))] = \mathbb{E}[exp(-\xi(f))].$$

Proof. I will be assuming this is true, as the proof involves a statement neither Dr. Hislop or I can figure out, which is the basis for 2. → 1. ☐

Step 2. Now, let ξ and $\xi(\Lambda, E)$ be Poisson point processes with intensities $\mu(dx) = n(E)dx$. We will show that these verify the properties of Lemma 5.

$$f_\zeta(x) = \frac{\tau}{(x-\sigma)^2 + \tau^2}$$

for some arbitrary $\zeta = \sigma + i\tau \in \mathcal{H}$, where $\mathcal{H}$ is the upper half plane. We now wish to analyze

$$\mathbb{E}[\xi(\Lambda, E)(f_\zeta)] = \mathbb{E}[\sum_j \frac{\tau}{(|\Lambda_L|(E_j(\Lambda_L) - E))^2 + \tau^2}]$$

$$= \frac{1}{|\Lambda_L|}\mathbb{E}[\sum_j \frac{|\Lambda_L|^{-1}\tau}{(E_j(\Lambda_L) - E + |\Lambda_L|^{-1}\sigma)^2 + (|\Lambda_L|^{-1}\tau)^2}]$$

$$= \frac{1}{|\Lambda_L|}\mathbb{E}[Tr(Im(G^{\Lambda_L}(E + |\Lambda_L|^{-1}\zeta)))]$$

$$= \frac{1}{\Lambda_L}\sum_{x\in\Lambda} \mathbb{E}[Im(\langle\delta_x, R^{\Lambda_L}(E + |\Lambda_L|^{-1}\zeta)\delta_x\rangle)].$$

Applying Lemma 1 to each term in the sum, we get if $(\langle\delta_x, R^{\Lambda_L}(E+|\Lambda_L|^{-1}\zeta)\delta_x\rangle)^{-1} = x+iy$ then

$$\mathbb{E}[Im(\langle\delta_x, R^{\Lambda_L}(E + |\Lambda_L|^{-1}\zeta)\delta_x\rangle)] = \mathbb{E}[\int_{-\infty}^{\infty} \frac{y}{(v-x)^2 + y^2}\rho(v)dv] \leq ||\rho||_\infty \int_{-\infty}^{\infty} f_\zeta(x)dx$$

which shows that

$$\mathbb{E}[\eta(\Lambda_L; E)(dx)] \leq ||\rho||_\infty dx$$

Similarly we can see for ξ that indeed this is true as well by Theorem 1.2.1 since

$$n(E) = \lim_{E_0\to E} \frac{N(E) - N(E_0)}{E - E_0} = \lim_{E\to E_0} \lim_{\Lambda_L\uparrow\mathbb{Z}^d} \frac{1}{|\Lambda_L|(E-E_0)}\mathbb{E}[Tr(P^{H_{\Lambda_L}}([E_0, E]))]$$

$$\leq \frac{||\rho||_\infty |\Lambda_L||E - E_0|}{|\Lambda_L|(E-E_0)} = ||\rho||_\infty.$$

Therefore we have that $n(E)dx \leq ||\rho||_\infty dx$

$$n(E)dx \leq ||\rho||_\infty dx$$

Thus, both $\eta(L, E)$ and ξ satisfy the conditions of the Lemma. Therefore it suffices to show that for $a_j > 0$, $j = 1, \dots n$ and $\zeta_j = \sigma_j + i\tau$ with $\tau > 0$ that

$$\lim_{L\to\infty} \mathbb{E}\big[exp\{-\frac{1}{|\Lambda_L|}\sum_j a_j Im(TrG^{\Lambda_L}(E + |\Lambda_L|^{-1}\zeta_j))\}\big] = \mathbb{E}[exp\{-\xi(\phi)\}] \tag{2.4}$$

Where

$$\phi(x) = \sum_j a_j \frac{\tau}{(x-\sigma_j)^2 + (\tau)^2}.$$

Step 3. We divide $[0,L]^d$ into $L^{\alpha d}$ cubes $\overline{\Lambda_p}$, $p = 1, \dots L^{\alpha d}$ with side length $L^{1-\alpha}$. We define

$$\Lambda_p = \overline{\Lambda_L} \cap \mathbb{Z}^d$$

We also choose the interior of Λ_p

$$int(\Lambda_p) = \{x \in \Lambda_p | \; dist(x, \partial\Lambda_p) > \beta ln(L)\}$$

And hence for any $z \in \mathbb{C}$ and any $x \in int(\Lambda_p)$ we can use our Geometric Resolvent Equation, Theorem 2.2.1

$$G^{\Lambda_L}(z;x,x) = G^{\Lambda_p}(z;x,x) + \sum_{(y,y')\in\partial\Lambda_p} G^{\Lambda_p}(z;x,y)G^{\Lambda_L}(z;y',x)$$

to get

$$|\frac{1}{|\Lambda_L|}\big(\text{Im Tr}G^{\Lambda_L}(z) - \sum_p \text{Im Tr}G^{\Lambda_p}(z)\big)| \le |\frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L} im(G^{\Lambda_L}(z;x,x)) - \frac{1}{|\Lambda_L|}\sum_p\sum_{x\in\Lambda_p} im(G^{\Lambda_p}(z;x,x))|$$

$$\le \frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L} |im(G^{\Lambda_p}(z;x,x))| + \frac{1}{|\Lambda_L|}\sum_{x\in\Lambda_L}\sum_{(y,y')\in\partial\Lambda_p} |G^{\Lambda_p}(z;x,y)||G^{\Lambda_L}(z;y',x)| - \frac{1}{|\Lambda_L|}\sum_p\sum_{x\in\Lambda_p}|G^{\Lambda_p}(z;x,x)|$$

$$\le \frac{1}{|\Lambda_L|}\sum_p \sum_{x\in\Lambda_p\setminus int(\Lambda_p)} im(G^{\Lambda_p}(z;x,x)) + im(G^{\Lambda_L}(z;x,x))$$

$$+\frac{1}{|\Lambda_L|}\sum_p\sum_{x\in int(\Lambda_p)}\sum_{(y,y')\in\partial\Lambda_p} |G^{\Lambda_p}(z;x,y)||G^{\Lambda_L}(z;y',x)|$$

$$= A_L + B_L$$

And hence we get A_L and B_L. We will use the fact that we are allowed to choose β and α to guarantee that $\mathbb{E}[A_L]$ and $\mathbb{E}[B_L]$ go to zero. For $\mathbb{E}[A_L]$ we can estimate since $\mathbb{E}[im(G^{\Lambda_p})(z;x,x)]$ and $\mathbb{E}[im(G^{\Lambda_L})(z;x,x)]$ are bounded uniformly by a constant (the same bound one gets from spectral averaging), we get

$$\mathbb{E}[A_L] = O(L^{-d}L^{\alpha d}L^{(1-\alpha)(d-1)}ln(L)) = O(L^{\alpha-1}ln(L)) \to 0$$

since $\alpha < 1$. Now for B_L we will use our localization estimate, and prove that the expectation of the fractional power of B_L goes to 0.

We will show that $\mathbb{E}[B_L^{\frac{s}{2}}] \to 0$ as $L \to \infty$. The reason this is enough is precisely a fact from real analysis. If $f_L(\lambda)$ is a function depdending on lambda, then if

$$\lim_{L\to\infty}\int_{\mathbb{R}} |f_L(\lambda)|d\mu(\lambda) = 0$$

then for μ almost every λ, we have $|f_L(\lambda)| \to 0$ μ almost everywhere. Then we will also have

$$|f_L(\lambda)|^p \to 0$$

for $p \geq 1$. Therefore analyzing $\mathbb{E}[B_L^{s/2}]$ we get

$$\mathbb{E}[B_L^{s/2}] \leq |\Lambda_L|^{-s/2} \sum_p \sum_{x \in int(C_p)} \sum_{(y,y') \in \partial\Lambda_p} \sqrt{E[|G^{\Lambda_p}(z;x,y)|^s]}\sqrt{E[|G^{\Lambda_L}(z;y;,x)|^s]}$$

so now once again we can use the fact that $E[|G^{\Lambda_L}(z;y;,x)|^s]$ is bounded by a uniform constant, and now we can use Theorem 2.1.3 to estimate

$$\mathbb{E}[B_L^{s/2}] \leq C|\Lambda_L|^{-s/2} \sum_p \sum_{x \in int(C_p)} \sum_{(y,y') \in \partial\Lambda_p} e^{-m|x-y|} = O(L^{d(2-\frac{s}{2})-1} L^{\alpha(1-d)} ln(L) L^{-m\beta})$$

Therefore if $\beta > \frac{1}{m}(d(2-s_0)-1-\alpha(d-1))$ we have that this term will go to 0 as $L \to \infty$. Thus in probability, $(B_L^{\frac{s}{2}})^{\frac{2}{s}} = B_L \to 0$ Therefore $A_L + B_L$ goes to 0 in probability uniformly for $\{|z-E| < r\} \cap \mathbb{H}$ So instead of 2.4 we can show

$$\lim_{L\to\infty} \mathbb{E}\big[exp\{-\frac{1}{|\Lambda_L|}\sum_p \sum_j a_j Im(TrG^{\Lambda_p}(E+|\Lambda_L|^{-1}\zeta_j))\}\big] = \mathbb{E}[exp\{-\xi(\phi)\}] \qquad (2.5)$$

We can rewrite 2.5, in terms of $\eta(L,E)$ that we had before. Therefore this can be rewritten as

$$\lim_{L\to\infty} \mathbb{E}[e^{-\eta(L,E)(\phi)}] = \mathbb{E}[e^{-\xi(\phi)}]$$

Thus our goal result is to prove the following proposition

Proposition 2. *As $L \to \infty$, $\eta(L,E)$ converges weakly to the Poisson point process with intensity measure $n(E)dx$.*

We have already shown that $\eta(\Lambda_p;E)$ is an asymptotically negligible array. Hence if we can show that

$$\sum_p \mathbb{P}(\eta(\Lambda_p;E)(A) \geq 1) \to n(E)|A| \qquad (2.6)$$

and

$$\sum_p \mathbb{P}(\eta(\Lambda_p;E)(A) \geq 2) \to 0 \qquad (2.7)$$

With the Minami estimate, it is easiest to show 2.7 via Theorem 1.3.1. This is because for each of these measures we can look at the trace of the spectral projector to get

$$\mathbb{P}(TrP^{|\Lambda_L|(H_{\Lambda_p}-E)}(A) \geq 2) \leq (||\rho||_\infty |A| L^{(\alpha-1)d})^2$$

and therefore the sum of all terms over p gives the estimate

$$\sum_p \mathbb{P}(\eta(\Lambda_p;E)(A) \geq 2) \leq (||\rho||_\infty |A| L^{-\alpha d})^2 L^{\alpha d} = O(L^{-\alpha d})$$

and therefore this goes to 0 as $L \to \infty$ for all borel sets A. The only thing left to show is (2.6). So to do this, consider f_ζ like before, then we get

$$\mathbb{E}[\eta(\Lambda_p;E)(f_\zeta)] = \frac{1}{|\Lambda_L|}\mathbb{E}\big[\sum_{x \in int(\Lambda_p)} + \sum_{x \in \Lambda_p \setminus int(\Lambda_p)} im(G^{\Lambda_p}(\lambda;x,x))\big]$$

where $\lambda = E + |\lambda_L|^{-1}\zeta$. Now in the same way we estimated A_L we get the second term in the sum being

$$O(L^{-d}L^{\alpha(d-1)}ln(L)) = O(L^{-\alpha d}) \to 0$$

Now for the first term, for $x \in int(\Lambda_p)$ one has the other geometric resolvent equation, Corollary 2,

$$G^{\Lambda_p}(z;x,x) = G(z;x,x) - \sum_{(y,y')\in\partial\Lambda_p} G^{\Lambda_p}(z;x,y)G(z;y',x)$$

And therefore we see that

$$\begin{aligned} &|\mathbb{E}[im(G^{\Lambda_p}(\lambda;x,x))] - \mathbb{E}[im(G(\lambda;x,x))] \\ &\leq \sum_{(y,y')\in\partial\Lambda_p} E[|G^{\Lambda_p}(\lambda;x,y)|G(\lambda;y',x)|] \\ &\leq (|\Lambda_L|/\tau)^{2-s} \sum_{(y,y')\in\partial\Lambda_p} \mathbb{E}[|G^{\Lambda_p}(\lambda;x,y)|^{\frac{s}{2}}|G(\lambda;y',x)|^{\frac{s}{2}}] \end{aligned}$$

Where we use Hölder's inequality with $s/2$ and $1 - s/2$ while the simple resolvent bound

$$|G(\lambda;x,y)| \leq \frac{1}{im(\lambda)}; \qquad |G^{\Lambda_L}(\lambda;y',x)| \leq \frac{1}{im(\lambda)}$$

for the $1 - \frac{s}{2}$ term. Therefore, if we use cauchy-schwartz in the sum term we get

$$\begin{aligned} &(|\Lambda_L|/\tau)^{2-s} \sum_{(y,y')\in\partial\Lambda_p} \sqrt{\mathbb{E}[|G^{\Lambda_p}(\lambda;x,y)|^s]}\sqrt{\mathbb{E}[|G(\lambda;y',x)|^s]} \\ &= O(L^{(3-s)d-1}L^{-\alpha(d-1)}L^{-m\beta}) \end{aligned}$$

if we once again use Theorem 2.1.3 to estimate this and this goes to zero if we choose β large enough, and this could be larger than our previous β. But, this means that independent of $x, y' \in \mathbb{Z}^d$ and $\lambda \in \mathbb{H}$, we get

$$\mathbb{E}[im(G^{\Lambda_p}(\lambda;x,x))] = \mathbb{E}[im(G(\lambda;x,x))] + o(1)$$

But now, we can use the fact that the expectation of our inner product with the resolvent is the integral of the resolvent with respect to the density of states measure which is absolutely continuous with respect to the lebesgue measure. And hence, after pulling the imaginary part out of the integral, we get

$$\mathbb{E}[im(G(\lambda;x,x))] = \int_{\mathbb{R}} im(\frac{1}{u-\lambda})n(u)du$$

which we get, with $\lambda = E + |\Lambda_L|^{-1}\zeta$

$$\int_{\mathbb{R}} n(u)\frac{|\Lambda_L|^{-1}\tau}{(u-(E+|\Lambda_L|^{-1}\sigma))^2 + (|\Lambda_L|^{-1}\tau)^2}du \to n(E)\pi$$

as $L \to \infty$ by the fact that we can change variables from $|\Lambda_L|^{-1}\sigma = x$ and $|\Lambda_L|^{-1}\tau = y$

and so we can transform this into

$$\lim_{(x,y)\to(0,0)} \int_{\mathbb{R}} n(u) \frac{y}{(u-E-x)^2+y^2} du$$

since $n(u) \in L^1(\mathbb{R})$, and

$$\frac{1}{\pi} \frac{y}{(u-E-x)^2+y^2}$$

is an approximation of the identity in y, and will converge to $\delta(u-E)$. Therefore we get

$$\mathbb{E}[\eta(\Lambda_p; E)(f_\zeta)] \sim \pi n(E) \frac{|int(\Lambda_p)|}{|\Lambda_L|} \sim \pi n(E) L^{-\alpha d}$$

So summing over p we get

$$\sum_p \mathbb{P}(\eta(\Lambda_p; E)(f_\zeta)) \geq 1) \to \sum_p L^{-\alpha d} \pi n(E) = \pi n(E)$$

Since one can estimate $\chi_A(x)$ by a linear combination of f_ζ's, we get

$$\sum_p \mathbb{P}(\eta(\Lambda_p; E)(A) \geq 1) \to n(E)|A|$$

which is what we wanted to show. Therefore we have that $\eta(L; E) \to \xi$ weakly to the Poisson point process with intensity measure $n(E)dx$. Thus, the proof is complete.

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