MA 614 – Enumerative Combinatorics[1]
Spring 2018

1. General Information

Dr. Benjamin Braun
Course Webpage: https://sites.google.com/view/braunmath/courses/ma-614-spring-2019
Email: benjamin.braun@uky.edu
Office Phone: 257-6810
Class Time/Location: 12:00-12:50PM, MWF, CB 335
Office Location/Hours: 831 POT, Mon 11:00-11:50, Wed 1:00-1:50, Fri 10:00-10:50, or by appointment. Feel free to knock if my door is open.

2. Texts

Primary Texts:

- *Enumerative Combinatorics, Volume 1, 2nd edition*, Richard Stanley.
- Course Notes, pdf available on website.

Additional Resources:

- *Enumerative Combinatorics, Volume 2*, Richard Stanley
- *The Two Cultures of Mathematics*, Timothy Gowers. Available electronically at

  www.dpmms.cam.ac.uk/∼wtg10/2cultures.pdf
- *A Course in Combinatorics*, Van Lint and Wilson, 2001.
- *A Course in Enumeration*, Martin Aigner, 2007.
- *A Walk Through Combinatorics: An Introduction to Enumeration and Graph Theory, 2nd edition*, by Miklos Bona, 2006.

3. Course Description

> Having vegetated on the fringes of mathematical science for centuries, combinatorics has now burgeoned into one of the fastest growing branches of mathematics . . . The mathematical world had been attracted by the success of algebra and analysis and only in recent years has it become clear . . . that combinatorics, the study of finite sets and finite structures, has its own problems and principles. These are independent of those in algebra and analysis but match them in difficulty, practical and theoretical interest, and beauty.
> László Lovász, *Combinatorial Problems and Exercises*

The basic problem of enumerative combinatorics is that of determining the number of elements of a finite set. This enterprise is interesting, subtle, surprising, and very challenging. Enumeration is part of the larger discipline of combinatorics and has connections to algebra, analysis, topology, number theory, probability, statistics, geometry, and a host of other areas in mathematics and science. Enumeration, and combinatorics in general, has a different feel than these other, more "traditional," areas of mathematics, a difference articulated by Timothy Gowers in the following passage.

> If the processes of abstraction and generalization, which are so important in mathematics, are of only limited use in combinatorics, then how can the subject be transmitted to future generations? One way of thinking about this question is to ask what the requirements of tomorrow's combinatorialists are likely to be . . . their priority is likely to be solving problems, so their interest in one of today's results

[1] I reserve the right to change or amend this syllabus at any time for any reason.

> will be closely related to whether, by understanding it, they will improve their own problem-solving ability. And this brings us straight to the heart of the matter. The important ideas of combinatorics do not usually appear in the form of precisely stated theorems, but more often as general principles of wide applicability.
> TIMOTHY GOWERS, *The Two Cultures of Mathematics*

Given this quality of combinatorial theory, the general student learning goals in this course are the following:

- To understand general principles of wide applicability for solving problems involving enumeration of finite sets, and
- To develop the ability to apply these principles to solve such problems.

We will develop a theory of enumeration based on common enumerative structures. Enumerative theory consists of understanding each of these common structures along with techniques for studying them.

Enumerative Structures and Principles

(1) Basic structures of enumeration
  (a) Fundamental integer sequences: Binomial coefficients, Catalan numbers, Fibonacci numbers, Eulerian numbers, Bell numbers, Stirling numbers.
  (b) Lattice paths.
  (c) Compositions and partitions.
  (d) Recurrences.
  (e) Permutations and permutation statistics.
  (f) q-analogues.
  (g) The twelvefold way.
(2) Generating functions
  (a) Ordinary generating functions.
  (b) Rational generating functions and partial fractions decompositions.
  (c) Exponential generating functions.
  (d) The exponential formula.
  (e) Tree enumeration and Cayley's theorem.
  (f) Lagrange inversion formula.
(3) Sieve Methods
  (a) The principle of inclusion-exclusion.
  (b) Unimodality and log-concavity of sequences.
  (c) The involution principle and sign-reversing involutions.
  (d) The Gessel-Viennot theorem.
(4) Partially Ordered Sets
  (a) Partially ordered sets (posets).
  (b) Lattices and their refinements.
  (c) The incidence algebra of a poset.
  (d) The zeta and Möbius functions for posets.
  (e) Computational methods for Möbius functions and the Möbius inversion formula.
  (f) Möbius algebras of lattices, Weisner's theorem, and the crosscut theorem.
  (g) Rank-selection, flag f- and h-vectors, and R-labelings.
  (h) Eulerian posets and duality.

4. Grading System

4.1. **Overview.** Your course grade will be determined by how well you demonstrate mastery of the *"Enumerative Structures and Principles"* listed previously. Here are the key features of the grading system:

(1) **There are no points.** All problems will be given one of two grades: "Pass" or "Revise". The grade for each problem will be based on a combination of (A) correctness of the solution and (B) quality of communication in your writing.
(2) **You can revise a significant amount of your work.** My goal is for you each to develop a profound understanding and mastery of enumerative combinatorics. To support this goal, you are allowed to **revise and resubmit up to two problems per week**, taken from any previous homework or exam.
(3) **The assignments are divided into collaborative and independent sections.** Mathematics is not completely collaborative, and not complete individual. Therefore, we must strive to develop our skills and abilities in both of these contexts. Your assignments will support this development.
(4) **All exams are take-home and have a time limit.** Each exam will be two hours long, but you will take them at a time and place of your choosing, outside of class.
(5) **You are expected to attend class.** You must be present at, prepared for, and engaged in class each day. If you need to miss class for some reason, please notify me ahead of time.

4.2. **Three Types of Homework Problems.**

4.2.1. *General information.*

- Homework will be due on Fridays. A partial selection of problems from each problem set will be graded.
- I strongly recommend you type your homework in Latex. Latex is a fantastic system for typesetting mathematics and is the standard typesetting method for professional mathematicians. If you have not used Latex before, I recommend you use overleaf.com to start.
- Searching the library or internet for solutions to problems is not allowed. The act of copying a written answer from another student and submitting it as your own will be considered cheating and will be dealt with according to University procedures.
- Each problem on the homework will be designated as either "collaborative" or "independent".

4.2.2. *Collaborative Problems.* With collaborative homework, you may collaborate with your classmates in developing ideas regarding homework problems; however, do not let cooperation degenerate into one person solving the problem and other people copying their answers. While it is important to celebrate mathematics as a social and cultural endeavor, it is also important that you work out the details for solutions on your own. You must write up your own answers to all the questions. **For each homework problem, indicate in your solution the people you shared ideas with.**

4.2.3. *Independent Problems.* With independent homework, you may *not* collaborate with your classmates at all. You are welcome to talk to your professor about your ideas and approaches to these problems, but not to your peers. **All revisions of independent homework should also be completed independently.**

4.2.4. *Problem-creation Problems.* You will be asked to create problems about the course material. You should have your own unique problems to submit, but you are allowed to brainstorm and consult with your peers about your problem ideas. These problems may be variations on homework problems or in-class examples you have seen previously, or variations on problems that other students share with you, or they may be problems of your original creation. For each of these problems, you should do the following:

- State the problem.
- Provide a correct and clearly-written solution to the problem.
- Write a short paragraph explaining how you created the problem, why you chose to submit it, and who you discussed your ideas with.

4.3. **Exams.** There will be one take-home midterm exam and a take-home cumulative final. Each exam should be taken during a continuous two-hour time period. You are expected to work on these completely independently and to follow all instructions regarding them. *You are not allowed to collaborate with other students on the exams.*

4.4. **Course Grades.** To earn a particular grade in the class, you need to complete all of the requirements in the column corresponding to that grade.

	A	B	C	D
Collaborative Problems	pass 90%	pass 80%	pass 70%	pass 60%
Independent Problems	pass 80%	pass 70%	pass 60%	pass 50%
Problem-creation Problems	pass 87%	pass 75%	none passed	none passed
Exam Problems	pass 50%	pass 40%	pass 30%	pass 20%

5. **Tentative Schedule**

For each day there is an assigned reading; you are expected to complete the reading for each day *before* class. In this schedule, "EC 2–5" refers to pages 2–5 of Stanley's Enumerative Combinatorics Vol 1, 2nd edition.

Sets, subsets, and integer compositions

Jan 9: Reading: Syllabus, EC 1–3
Discussion: Introduction to counting

Jan 11: Reading: EC 13–17 (start at bottom paragraph on page 13, OMIT examples 1.1.16 and 1.1.17)
Discussion: Pascal's triangle, the binomial recurrence, and the binomial theorem
HW # 1 due

Jan 14: Reading: EC 17–19 (only first half of 19, stop after the second direct proof).
Discussion: Binomial coefficients and integer compositions; multisets.

Jan 16: Reading: EC 3–8
Discussion: Generating functions, the binomial series, applications to binomial coefficients and compositions
HW # 2 due

Jan 18: Reading: Wikipedia pages on Fibonacci Numbers and Fibonacci polynomials:
http://en.wikipedia.org/wiki/Fibonacci_number (just skim for an overview)
http://en.wikipedia.org/wiki/Fibonacci_polynomials
Discussion: Compositions using 1's and 2's, Fibonacci numbers, Fibonacci polynomials.

Variations on integer compositions

Jan 21: MLK Holiday – no class

Jan 23: Reading: EC 10 (only Example 1.1.12), EC 464–466 (stop after Example 4.1.2)
Discussion: Recurrence relations and ordinary generating functions, Binet's formula for Fibonacci numbers

Jan 25: Reading: EC 466–467 (only Example 4.1.3)
Discussion: Combinatorial interpretations for linear recurrence relations
HW # 3 due

Jan 28: Reading: EC 58, 61–62 (stop at Prop 1.8.1), 63–64 (start with Prop 1.8.4, stop after the first proof of Prop 1.8.5)
Discussion: Partition Identities

The twelvefold way

Jan 30: Reading: EC 71–75, on page 75 stop after the proof of Entry 3.
Discussion: The twelvefold way, Bell numbers, and Stirling numbers of the second kind.

Feb 1: Reading: EC 75–76 and 79–80 (on page 79, start with the proof of Entry 4)
Discussion: The twelvefold way, continued.
HW # 4 due

Permutations

Feb 4: Reading: EC 20 (only definition of $\mathfrak{S}_S$ at bottom), EC 22–24 (stop after proof of Theorem 1.3.2)
Discussion: The Symmetric Group $\mathfrak{S}_n$, word/two-line notation and cycle notation, and cycle structure

Feb 6: Reading: EC 26–29 (for Prop 1.3.7, read *only* the first proof and fourth proof)
Discussion: (signless) Stirling numbers of the first kind.

Feb 8: Reading: EC 29–31 (read the entire subsection regarding inversions), EC 42–43 (last paragraph of 42, first paragraph of 43).
Discussion: Inversions and diagrams of permutations.
HW # 5 due

Feb 11: Reading: EC 31 (read only the definition of $D(w)$), EC 32–34 (start at last paragraph on 32, read entire proof of Prop 1.4.5)
Discussion: Descent sets and Eulerian numbers

Multinomial coefficients, lattice paths, and Catalan numbers

Feb 13: Reading: EC 20–22
Discussion: Multinomial coefficients and lattice paths.

Feb 15: Reading: http://en.wikipedia.org/wiki/Catalan_number, read the sections titled "Properties," "Applications in Combinatorics," and the *second proof* in the section "Proof of the formula."
Discussion: Catalan numbers, Dyck paths, the Catalan recurrence.
HW # 6 due

Feb 18: Reading: http://en.wikipedia.org/wiki/Catalan_number, first proof in the section "Proof of the formula", and all of http://en.wikipedia.org/wiki/Narayana_number
Discussion: Catalan numbers, generating functions, and Narayana numbers

q-analogues

Feb 20: Reading: EC 54–58 (start at section 1.7, *skip both proofs* of Prop 1.7.1 as we will prove a special case in class, read Prop 1.7.2 and its proof).
Discussion: q-multinomial polynomials, Gaussian polynomials, and multiset permutation inversions.

Feb 22: Reading: EC 59–60, omit proof of Prop 1.7.3 (we will give a straightforward proof in class).
Discussion: Gaussian polynomials, partitions, and lattice paths.
HW # 7 due

Exponential generating functions

Feb 25: Reading: None.
Discussion: Multiplication of exponential generating functions, derangements, and involutions.
EXAM: Distribute take-home midterm exam. The midterm exam covers material up to and including Feb 18.

Feb 27: Reading: None.
Discussion: The Block-Partitioned Structure Principle and the exponential formula.
Mar 1: Reading: None.
Discussion: Applications of the exponential formula
HW# 8 due

Mar 4: Reading: None.
Discussion: Recurrences for EGFs, permutations with restricted cycle types.
EXAM: Midterm exam due.
Mar 6: Reading: None.
Discussion: The Point-Removal Structure Principle and the Lagrange Inversion Formula.
Mar 8: Reading: None.
Discussion: Tree enumeration, applications of point-removal and Lagrange inversion.
HW # 9 due

Spring Break: March 11–15 – no class

Inclusion-exclusion and sieve methods

Mar 18: Reading: http://en.wikipedia.org/wiki/Inclusion-exclusion_principle, read the "statement" section and the first two examples in the "Examples" section. NOTE: This is equivalent to section 2.1 in EC.
Discussion: The principle of inclusion-exclusion.
Mar 20: Reading: EC 212-213, and Exercise 1.50 (on page 213 stop at the section that is "a more complicated example" and read the Exercise as if the problems were theorem statements).
Discussion: Sign-reversing involutions, unimodal sequences, and log-concave sequences.
Mar 22: Reading: EC 215–218 (start at section 2.7, skip the proof of Theorem 2.7.1 as we will prove a special case in class).
Discussion: The Gessel-Viennot theorem.
HW # 10 due

Posets and lattices

Mar 25: Reading: EC 241–243 (stop at definition of isomorphic).
Discussion: Posets.
Mar 27: Reading: EC 243–246 (stop at section 3.2).
Discussion: Posets.
Mar 29: Reading: EC 246–247.
Discussion: New posets from old.
HW # 11 due

Apr 1: Reading: EC 248–252 (read all of section 3.3).
Discussion: Lattices.
Apr 3: Reading: EC 252–253 (stop after proof of Theorem 3.4.1).
Discussion: Finite distributive lattices.

Incidence algebras, zeta functions, and Möbius functions

Apr 5: Reading: EC 261–263 (stop after discussion of $(\zeta - 1)$).
Discussion: The incidence algebra of a poset and the zeta function.
HW # 12 due

Apr 8: Reading: EC 264–265 and 263 and 268-269 (on 263 only read the first justification regarding $2-\zeta$ and on 268-269 only read Prop 3.8.5 and its proof).
Discussion: Möbius functions and the Möbius inversion formula.

Apr 10: Reading: EC 266–267 (stop after Example 3.8.3), EC 274-275 (start at section 3.9, stop just before Theorem 3.9.2).
Discussion: Techniques of Computation: the product theorem, Weisner's theorem and the crosscut theorem.

Apr 12: Reading: EC 275–276 (start with Cor 3.9.3).
Discussion: Applications of Weisner's theorem.
HW #13 due

Apr 15: Reading: None.
Discussion: Möbius algebras for lattices

Apr 17: Reading: EC 277–280.
The Möbius function of a semi-modular lattice.

Properties inspired by polytopes and topology: rank selection, flag f and h vectors, R-labeling, and Eulerian posets

Apr 19: Reading: EC 257–258 (read only the definition of *linear extension*), 293–295 (skip the NOTE regarding simplicial complexes, and skip Theorem 3.13.3).
Discussion: Rank selection and the flag f- and h-vector of a graded poset.
HW # 14 due

Apr 22: Reading: None.
More on Rank selection.

Apr 24: Reading: EC 295–298.
Discussion: R-labelings.

Apr 26 Reading: EC 310 and 311–312 (on 310 only read the definition of Eulerian at the top and on 311–312 only read from Lemma 3.16.3 through Cor 3.16.6).
Discussion: Eulerian posets and duality
HW # 15 due
EXAM: Final exam distributed – due Tuesday, April 30, 5PM, to my office.

6. Administrative Policies

6.1. **Excused Absences.** Students need to notify the professor of absences prior to class when possible. Senate Rules 5.2.4.2 defines the following as acceptable reasons for excused absences: (a) serious illness, (b) illness or death of family member, (c) University-related trips, (d) major religious holidays, and (e) other circumstances found to fit reasonable cause for nonattendance by the professor.

Students anticipating an absence for a major religious holiday are responsible for notifying the instructor in writing of anticipated absences due to their observance of such holidays no later than the last day in the semester to add a class. Two weeks prior to the absence is reasonable, but should not be given any later. Information regarding major religious holidays may be obtained through the Ombud (859-257-3737, http://www.uky.edu/Ombud/ForStudents_ExcusedAbsences.php.

Students are expected to withdraw from the class if more than 20% of the classes scheduled for the semester are missed (excused) per University policy.

Per Senate Rule 5.2.4.2, students missing any graded work due to an excused absence are responsible: for informing the Instructor of Record about their excused absence within one week following the period of the excused absence (except where prior notification is required); and for making up the missed work. The professor must give the student an opportunity to make up the work and/or the exams missed due to an excused absence, and shall do so, if feasible, during the semester in which the absence occurred.

6.2. **Verification of Absences.** Students may be asked to verify their absences in order for them to be considered excused. Senate Rule 5.2.4.2 states that faculty have the right to request appropriate verification when students claim an excused absence because of illness, or death in the family. Appropriate notification of absences due to University-related trips is required prior to the absence when feasible and in no case more than one week after the absence.

6.3. **Academic Integrity.** Per University policy, students shall not plagiarize, cheat, or falsify or misuse academic records. Students are expected to adhere to University policy on cheating and plagiarism in all courses. The minimum penalty for a first offense is a zero on the assignment on which the offense occurred. If the offense is considered severe or the student has other academic offenses on their record, more serious penalties, up to suspension from the University may be imposed.

Plagiarism and cheating are serious breaches of academic conduct. Each student is advised to become familiar with the various forms of academic dishonesty as explained in the Code of Student Rights and Responsibilities. Complete information can be found at the following website: http://www.uky.edu/Ombud. A plea of ignorance is not acceptable as a defense against the charge of academic dishonesty. It is important that you review this information as all ideas borrowed from others need to be properly credited.

Senate Rules 6.3.1 (see http://www.uky.edu/Faculty/Senate/ for the current set of Senate Rules) states that all academic work, written or otherwise, submitted by students to their instructors or other academic supervisors, is expected to be the result of their own thought, research, or self-expression. In cases where students feel unsure about a question of plagiarism involving their work, they are obliged to consult their instructors on the matter before submission.

When students submit work purporting to be their own, but which in any way borrows ideas, organization, wording, or content from another source without appropriate acknowledgment of the fact, the students are guilty of plagiarism.

Plagiarism includes reproducing someone else's work (including, but not limited to a published article, a book, a website, computer code, or a paper from a friend) without clear attribution. Plagiarism also includes the practice of employing or allowing another person to alter or revise the work, which a student submits as his/her own, whoever that other person may be. Students may discuss assignments among themselves or with an instructor or tutor, but when the actual work is done, it must be done by the student, and the student alone.

When a student's assignment involves research in outside sources or information, the student must carefully acknowledge exactly what, where and how he/she has employed them. If the words of someone else are used, the student must put quotation marks around the passage in question and add an appropriate indication of its origin. Making simple changes while leaving the organization, content, and phraseology intact is plagiaristic. However, nothing in these Rules shall apply to those ideas, which are so generally and freely circulated as to be a part of the public domain.

Please note: Any assignment you turn in may be submitted to an electronic database to check for plagiarism.

6.4. **Accommodations due to disability.** If you have a documented disability that requires academic accommodations, please see me as soon as possible during scheduled office hours. In order to receive accommodations in this course, you must provide me with a Letter of Accommodation from the Disability Resource Center (DRC). The DRC coordinates campus disability services available to students with disabilities. It is located on the corner of Rose Street and Huguelet Drive in the Multidisciplinary Science Building, Suite 407. You can reach them via phone at (859) 257-2754 and via email at drc@uky.edu. Their web address is http://www.uky.edu/DisabilityResourceCenter.

6.5. **Non-Discrimination Statement and Title IX Information.** The University of Kentucky faculty are committed to supporting students and upholding the University's non-discrimination policy.

Discrimination is prohibited at UK. If you experience an incident of discrimination we encourage you to report it to Institutional Equity & Equal Opportunity (IEEO) Office, 13 Main Building, (859) 257-8927.

Acts of Sex- and Gender-Based Discrimination or Interpersonal Violence: If you experience an incident of sex- or gender-based discrimination or interpersonal violence, we encourage you to report it. While you may talk to a faculty member or TA/RA/GA, understand that as a "Responsible Employee" of the University these individuals MUST report any acts of violence (including verbal bullying and sexual harassment) to the University's Title IX Coordinator in the IEEO Office. If you would like to speak with someone who may be able to afford you confidentiality, the Violence Intervention and Prevention (VIP) program (Frazee Hall Lower Level; http://www.uky.edu/StudentAffairs/VIPCenter/), the Counseling Center (106 Frazee Hall, http://www.uky.edu/S and the University Health Services (http://ukhealthcare.uky.edu/uhs/student-health/) are confidential resources on campus.