Problem AC1: Suppose the $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a (not necessarily complete) metric space X. Show that $\{d(p_n, q_n)\}$ has a limit as $n \to \infty$.

Proof. Claim: $\{d(p_n, q_n)\}$ is a Cauchy sequence of real numbers.
Let $m, n \in \mathbb{N}$. Consider $d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n)$. Therefore we have

$$d(p_n, q_n) - d(p_m, q_m) \leq d(p_n, p_m) + d(q_m, q_n). \tag{1}$$

Similarly, $d(p_m, q_m) \leq d(p_m, p_n) + d(p_n, q_n) + d(q_n, q_m)$. If we manipulate this inequality, we have

$$d(p_n, q_n) - d(p_m, q_m) \geq -\big(d(p_n, p_m) + d(q_n, q_m)\big). \tag{2}$$

Equations (1) and (2) give us that

$$0 \leq |d(p_n, q_n) - d(p_m, q_m)| \leq d(p_n, p_m) + d(q_n, q_m).$$

Since $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences, if we let $m, n \to \infty$, then

$$|d(p_n, q_n) - d(p_m, q_m)| \to 0 \quad as \quad m, n \to \infty.$$

Hence $\{d(p_n, q_n)\}$ is Cauchy.
Since $\mathbb{R}$ is complete, $\lim_{n\to\infty} d(p_n, q_n)$ exists. □

Problem AC2: Suppose $f : (0, 1) \to \mathbb{R}$ has the property that

$$|f(x) - f(y)| \leq |x - y| \quad \forall x, y \in (0, 1).$$

Show $\lim_{x\to 0^+} f(x)$ exists.

Proof. By definition of infimum of $(0, 1)$, we can create a sequence $\{x_n\} \subset (0, 1)$ so that $x_n \to 0$. Therefore this sequence is Cauchy. Let $\varepsilon > 0$. Then there is an N so that $|x_n - x_m| < \varepsilon$ when $m, n \geq N$.
By the given property, for this same N, $|f(x_n) - f(x_m)| \leq |x_n - x_m| < \varepsilon$ when $m, n \geq N$. Therefore $\{f(x_n)\}$ is a Cauchy sequence of real numbers and hence is convergent. That is $\lim_{x\to 0^+} f(x) = \lim_{n\to\infty} f(x_n)$ exists. □

Problem AC3: Suppose $f : [0, 1] \to \mathbb{R}$ has the property that

$$|f(x) - f(y)| \leq \frac{1}{2}|x - y| \quad \forall x, y \in [0, 1].$$

Show that there is exactly one fixed point in $[0, 1]$.

Proof. As stated, the claim is false. For example, let $f(x) = -\frac{1}{2}x + 4$. Then f satisfies the property but yet has NO fixed points in $[0, 1]$. □

Problem AC4: Suppose $\{f_n\}$ is a decreasing sequence of nonnegative continuous functions on $[0,1]$, i.e., $0 \leq ... \leq f_{n+1}(x) \leq f_n(x) \leq ...$ for each $x \in [0,1]$. Suppose further that $\lim_{n\to\infty} f_n(x) = 0$ for each x. Show that $f_n \to 0$ uniformly in $[0,1]$.

Proof. Let $\varepsilon > 0$. Define for each n, $E_n := \{x \in [0,1] : f_n(x) < \epsilon\}$.
Claim 1: For each n, E_n is open.
Notice, for each n, $E_n = f_n^{-1}([0,\varepsilon)) = f_n^{-1}\big((-\varepsilon,\varepsilon)\big)$. These are open since $(-\varepsilon,\varepsilon)$ is open and f_n is continuous for each n.
Claim 2: For each n, $E_n \subset E_{n+1}$.
This is true since the sequence is decreasing.
Claim 3: $[0,1] \subset \bigcup_{n=1}^{\infty} E_n$.
This is true since, for each $x \in [0,1]$, $f_n(x) \to 0$ as $n \to \infty$.
Claims 1 and 3 give us that $\{E_n\}$ is an open cover of $[0,1]$. Since $[0,1]$ is compact, there is a finite subcover $\{E_{n_k}\}_{k=1}^m$ of $[0,1]$. Let $N = max\{n_1, ..., n_m\}$. Then claim 2 and the fact that this finite subcollection covers $[0,1]$ gives us that for $n \geq N$, $[0,1] \subset E_N \subset E_n$.
That is there is an N so that for $n \geq N$, $f_n(x) < \varepsilon$ for all $x \in [0,1]$. Hence uniform convergence to 0. □

Problem RA1: Suppose $f : \mathbb{R} \to \mathbb{R}$ is integrable. Show that for any $\varepsilon > 0$, there exists a bounded function, g, of compact support such that

$$\int_{\mathbb{R}} |f - g| < \varepsilon.$$

Proof. Let $\varepsilon > 0$. For $n \in \mathbb{N}$, define

$$f_n(x) = \begin{cases} f(x) & if \quad |x| \leq n \quad and \quad |f(x)| \leq n \\ 0 & else \end{cases}.$$

Notice $|f_n| \leq |f|$ for any n and $f_n \to f$ point-wise and f is integrable. Therefore by the LDCT, there is an N so that

$$\int_{\mathbb{R}} |f - f_N| < \varepsilon.$$

So $g = f_N$. □

Problem RA2: Suppose $A \subset \mathbb{R}$ is measurable, and for any open interval (a,b),

$$m(A \cap (a,b)) \leq \frac{1}{2}(b-a).$$

Show that A has measure zero.

Proof. Method 1: Since A is measurable, almost every point in A is of Lebesgue density in A. That is

$$\lim_{t>0, t\to 0} \frac{m((x-t,x+t)\cap A)}{m((x-t,x+t))} = 1 \quad a.e. \quad x \in A.$$

Then the points in A that do not satisfy this form a subset of measure zero.
However, thanks to the property on A, for every $x \in A$ and for any $t > 0$, $\dfrac{m((x-t,x+t)\cap A)}{m(x-t,x+t)} \leq \dfrac{1}{2} < 1$. Therefore the limit equation will never be satisfied by any $x \in A$. Then A is that subset of measure zero.

Method 2: Recall that any open set can be represented as an countable union of disjoint open intervals. Let $\varepsilon > 0$. Since A is measurable, there exists and open set $\mathcal{O} = \bigcup_{j=1}^{\infty}(a_j, b_j)$, where each interval is disjoint, so that $A \subset \mathcal{O}$ and

$$m(\mathcal{O}) \leq m(A) + \varepsilon = m(\bigcup_j((a_j,b_j)\cap A)) + \varepsilon = \sum_j m((a_j,b_j)\cap A) + \varepsilon$$
$$\leq \frac{1}{2}\sum_j m((a_j,b_j)) + \varepsilon = \frac{1}{2}m(\mathcal{O}) + \varepsilon.$$

Since ε is arbitrary, we have $m(\mathcal{O}) \leq \frac{1}{2}m(\mathcal{O})$. This can only happen if $m(\mathcal{O}) = 0$ and by monotonicity of measure, $m(A) = 0$. □

Problem RA3: Consider the following statements:
(a) $f \in L^1(\mathbb{R})$,
(b) $\exists C$ so that $m(\{x \in \mathbb{R} : |f(x)| > t\}) \leq Ct^{-1}$ for all $t > 0$.
Give a proof that (a) implies (b), and give an example to show (b) does not imply (a).

Proof. ($\Rightarrow$) Let $t > 0$. Let $E_t = \{x : |f(x)| > t\}$. Then $m(E_t) = \int_{E_t} 1dx \leq t^{-1}\int_{E_t} f(x)dx \leq t^{-1}\int_{\mathbb{R}} f(x)dx$. Since f is integrable, $C = \int_{\mathbb{R}} f(x)dx < \infty$.
($\Leftarrow$) Consider f so that $f(x) = \frac{1}{x}$ for $x > 0$ and $f(x) = 0$ for $x \leq 0$. Then when $C = 1$, (b) is satisfied. But yet f is not integrable. □

Problem RA4:
(a) Suppose E is a Lebesgue measurable set of measure zero and $B \subset E$. Prove that B has measure zero.

Proof. Um, monotonicity of measure? □

(b) Suppose that E is a Lebesgue measurable set. Suppose that there is a nonnegative measurable function f with $f(x) > 0$ a.e and f is integrable on E. Show that if $\int_E f = 0$, then $m(E) = 0$.

Proof. Since Lebesgue integral ignores measure zero, we can proceed as if $f > 0$ on all of E. Since $f > 0$ is measurable on E, by the simple approximation theorem, there is an increasing sequence of nonnegative simple functions on E converging pointwise to f. Since f is integrable, we can use LDCT (really, we may only need monotone convergence theorem). Either way let $\varepsilon > 0$ so that there is a positive simple function, g, so that $\int_E |g - f| < \varepsilon$. Then we have

$$\int_E g \leq \int_E |g - f| + \int_E |f| = \int_E |g - f| < \varepsilon.$$

Since g is simple $g(x) = \sum_{k=1}^{N} c_k \chi_{E_k}$, where $\{E_k\}$ are disjoint subsets of E.
So we have $\sum_k c_k m(E_k) < \varepsilon$. Since we can make ε arbitrarily smaller, we have $\sum_k c_k m(E_k) = 0$. By how we found g and since each c_k is distinct and positive, the only way this equality can happen is if each $m(E_k) = 0$. Hence by additivity of measure, $m(E) = 0$. □