Problem AC 1: Suppose $A, B \subset \mathbb{R}$ are nonempty sets which are bounded above. Let

$$(A+B) = \{a+b : a \in A, b \in B\}.$$

Show that

$$\sup(A+B) = \sup A + \sup B.$$

Proof. Let $\alpha = \sup A$, $\beta = \sup B$, $\gamma = \sup(A+B)$. Let $a \in A$ and $b \in B$.
Then

$$\begin{aligned} & a+b-b = a \le \alpha \\ \Rightarrow\ & a+b-\alpha \le b \le \beta \\ \Rightarrow\ & a+b \le \alpha+\beta \\ \Rightarrow\ & \gamma \le \alpha+\beta. \end{aligned}$$

Similarly,

$$\begin{aligned} & a+b \le \gamma \\ \Rightarrow\ & a \le \alpha \le \gamma - b \\ \Rightarrow\ & b \le \beta \le \gamma - \alpha \\ \Rightarrow\ & \alpha+\beta \le \gamma. \end{aligned}$$

□

Problem AC 2: Suppose f is Riemann integrable on $[a,b]$ so that there is a function g continuous on $[a,b]$ with $f = g'$ for all $x \in (a,b)$. Show that

$$\int_a^b f(x)dx = g(b) - g(a).$$

Remark: It does not say f is continuous. So we cannot use FTC.

Proof. Let $\varepsilon > 0$. Since f is Riemman integrable, there is a partition, $P = \{x_0 = a, x_1, ..., x_n = b\}$, of $[a,b]$ so that

$$\Big| \int_a^b f(x)dx - \sum_{k=1}^{n} f(t_k)(x_k - x_{k-1}) \Big| < \varepsilon, \quad \forall t_k \in [x_{k-1}, x_k]. \tag{1}$$

By the Mean Value Theorem, for any k, there is $c_k \in (x_{k-1}, x_k)$ so that

$$f(c_k) = \frac{g(x_k) - g(x_{k-1})}{x_k - x_{k-1}}. \tag{2}$$

So by (1) and (2), we have

$$\begin{aligned} \Big| \int_a^b f(x)dx - (g(b)-g(a)) \Big| &= \Big| \int_a^b f(x)dx - \sum_{k=1}^{n} (g(x_k) - g(x_{k-1})) \Big| \\ &= \Big| \int_a^b f(x)dx - \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1}) \Big| < \varepsilon\,. \end{aligned}$$

□

Problem AC 3: Suppose f is differentiable on $\mathbb{R}$. We say that f has the *reverse mean value property* if for every $c \in \mathbb{R}$, there exists $a, b \in \mathbb{R}$ with $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Is it true that every continuously differentiable function has the reverse mean value property?

Soln. False. Consider $f(x) = x^3$ at $c = 0$. The slope between any two points properly containing 0 is always positve, while $f'(0) = 0$. Also f is continuously differentiable. □

Problem AC 4: Suppose $\{a_n\}$ is a positive sequence such that $\lim_{n\to\infty} a_n = 0$. Show that there exists a positive sequence $\{b_n\}$ such that $\lim_{n\to\infty} b_n = 0$ and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0.$$

Soln. Let $b_n = \sqrt{a_n}$. □

Problem RA 1: Let $A = [0, 2]$ and $f \in L^1(A)$. Define $f_n : A \to \mathbb{R}$ by

$$f_n(x) = x^{1/n} f(x), \quad n \geq 1.$$

Show that $f_n \in L^1(A)$, and find $\lim_{n\to\infty} \int_A f_n$.

Proof. Since f is integrable on A, $2f$ is integrable on A.
Note $|f_n| \leq 2|f|$. So by monotonicity of integration, f_n is integrable on $[a, b]$ for each n.
Also note that $f_n(x) \to f(x)$ a.e on A (0 is the only point in which it doesn't). Therefore by LDCT,

$$\lim_{n\to\infty} \int_A f_n = \int_A f.$$

□

Problem RA 2: Let $A \subset \mathbb{R}^n$ be a set of finite measure, and $f : A \to \mathbb{R}$ be measurable and finite-valued. Suppose that

$$F : A \times A \to \mathbb{R}, \quad F(x, y) = f(x) + f(y)$$

is integrable on $A \times A$. Let μ denote Lebesgue measure on $A \times A$. Show that f is integrable on A.

Proof. First note that if $m(A) = 0$, then the result holds trivially.
Assume $0 < m(A) < \infty$. By Fubini,

$$\int_A \left(\int_A \big|f(x) + f(y)\big| dx \right) dy = \int_{A \times A} \big|F(x, y)\big| d\mu < \infty.$$

Therefore $\int_A \big|f(x) + f(y)\big| dx < \infty$ for a.e. $y \in A$. Let y_0 be such an element in A.
Since f is finite-valued and A has finite measure, $\int_A |f(y_0)| dx = |f(y_0)| m(A) < \infty$.
Therefore

$$\int_A |f(x)| dx = \int_A |f(x) + f(y_0) - f(y_0)| dx \leq \int_A |f(x) + f(y_0)| dx + \int_A |f(y_0)| dx < \infty.$$

□

Problem RA 3: Let $f : [0, 1] \to (0, \infty)$ be an absolutely continuous function. Show that $\frac{1}{f}$ is absolutely continuous on $[0, 1]$.

Proof. First, since f is absolutely continuous on $[0, 1]$, f is continuous on $[a, b]$. Also f is positive valued. Therefore f attains a positive minimum, $m \leq f(x)$ for all $x \in [a, b]$.
Let $\varepsilon > 0$. With f absolutely continuous, let $\delta > 0$ correspond to $m^2 \varepsilon > 0$. Now let $\{[a_k, b_k]\}_{k=1}^N$ be a collection of disjoint closed subintervals of $[a, b]$ so that $\sum_{k=1}^N b_k - a_k < \delta$. Then

$$\sum_{k=1}^N \left| \frac{1}{f(a_k)} - \frac{1}{f(b_k)} \right| = \sum_{k=1}^N \frac{|f(a_k) - f(b_k)|}{f(a_k) f(b_k)} \leq \frac{1}{m^2} \sum_{k=1}^N |f(a_k) - f(b_k)| < \frac{1}{m^2} m^2 \varepsilon = \varepsilon\,.$$

Let $\frac{1}{f}$ is absolutely continuous on $[a, b]$. □

Problem RA 4: Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is measurable, and $\int_E f = 0$ for every measurable set E. Show that $f(x) = 0$ a.e.

Proof. Let

$$Z_1 = \{x \in \mathbb{R}^n : f(x) > 0\}$$
$$Z_2 = \{x \in \mathbb{R}^n : f(x) < 0\}.$$

Note first that both Z_1 and Z_2 are measurable since f is measurable.
Note that since f is positive on Z_1 there is a increasing sequence of positive simple functions that converges pointwise on a.e. on Z_1. Therefore by MCT, given any $\varepsilon > 0$, there is a simple function, $\varphi = \sum_{k=1}^{m} c_k \chi_{Z_{1,k}}$, from the sequence so that

$$\sum_{k=1}^{m} c_k m(Z_{1,k}) = \left|\left(\int_{Z_1} \varphi\right) - 0\right| = \left|\int_{Z_1} \varphi - \int_{Z_1} f\right| < \varepsilon\,.$$

Since we can take ε arbitrarily small, $\sum_{k=1}^{m} c_k m(Z_{1,k}) = 0$. Since φ is positive, the coefficients must all be positive. Therefore $m(Z_{1,k}) = 0$ for each k. Then $m(Z_1) = \sum_{k=1}^{m} m(Z_{1,k}) = 0$.
Let $g(x) = -f(x)$. Notice g is measurable, $\int_E g = 0$ for every measurable E, and $Z_2 = \{x \in \mathbb{R}^n : g(x) > 0\}$. Now repeat the argument above with g and we get $m(Z_2) = 0$.
Therefore $f(x) = 0$ a.e. □