Problem AC 1: Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and that

$$\lim_{n\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0.$$

Prove that f is uniformly continuous.

Proof. Look to **Problem AC 2** on January 2020 prelim solutions. $\square$

Problem AC 2: Suppose $\{I_k\}_{k\in\mathbb{N}}$ is a sequence of closed intervals $I_k = [a_k, b_k]$, such that $I_{k+1} \subseteq I_k$. Show that

$$C = \bigcap_{k=1}^{\infty} I_k$$

is non-empty. Must this be true if the I_k are open?

Proof. First note that for any $i, j \in \mathbb{N}$, $a_i \leq b_j$. If $i \leq j$, then $a_i \leq a_j \leq b_j$. If $i > j$, then $a_i \leq b_i \leq b_j$.
Therefore $a = sup\{a_k\}$ exists and $a \leq b_j$ for each j.
For each $n \in \mathbb{N}$, there is a_{k_n} so that $a - \frac{1}{n} < a_{k_n} \leq a$. So $a_{k_n} \nearrow a$.
Let $k \in \mathbb{N}$. There is N so that $k_n \geq k$ for each $n \geq N$. Therefore $a_k \leq a_{k_n} \leq a \leq b_{k_n} \leq b_k$ for each $n \geq N$. Since I_k closed, $a, a_k, b_k \in I_k$.
Therefore $a \in I_k$ for each k. So $a \in C$, and thus C is non-empty.
However if the I_k are open, this is not necessarily true. For example, if for each $k \in \mathbb{N}$, $I_k = (1 - \frac{1}{k}, 1)$. Then C empty. $\square$

Problem AC 3: Suppose $\{a_n\}$ is a sequence of positive numbers such that the limit

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

exists and is less than 1. Show *directly* that

$$\sum_{n=1}^{\infty} a_n$$

converges.

Proof. First note that since the sequence is positive, L is nonnegative.
Second if we take $r \in (L, 1)$ then there is N so that for $n > N$, $\dfrac{a_{n+1}}{a_n} < r$. Then for each

$k \in \mathbb{N}$, $a_{N+k} < r^k a_N$. Also, since $0 < r < 1$, $\sum_{k=1}^{\infty} a_N r^k < \infty$.
Therefore

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{k=1}^{\infty} a_{N+k} < \sum_{n=1}^{N} a_n + \sum_{k=1}^{\infty} a_N r^k < \infty.$$

□

Problem AC 4: Consider the sequence defined recursively by $a_1 = 1$ and $a_{n+1} = \sqrt{2a_n}$ for $n \geq 1$. Show that this sequence converges, and determine the limit.

Proof. Claim: $a_n \leq a_{n+1} \leq 2$, for each n.
We will use induction to show $a_n \leq a_{n+1}$. We can see $a_1 = 1 \leq \sqrt{2} = a_2$. Suppose this is true for $n \geq 1$. Consider $a_{n+2} = \sqrt{2a_{n+1}} \geq \sqrt{2a_n} = a_{n+1}$.
Now we will use induction to show $a_n \leq 2$ for each n. Again we see $a_1 = 1 \leq 2$. Suppose this is true for $n \geq 1$. Then $a_{n+1} = \sqrt{2a_n} \leq \sqrt{4} = 2$.
With this claim, the sequence converges to some $a \in \mathbb{R}$ by monotone convergence theorem for sequences. Now what is the limit? Note

$$a = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2a_n} = \sqrt{2a}.$$

If we solve for a and recall that everything in the sequence and hence a is greater than 0, we find $a = 2$. □

Problem RA 1: Let $f(x, y)$ be a continuous function in $\mathbb{R}^2$. Assume that $\partial_x f$ exists and is continuous in $\mathbb{R}^2$. Let

$$g(x) = \int_0^1 f(x, y)dy.$$

Show that g is differentiable and

$$g'(x) = \int_0^1 \partial_x f(x, y)dy.$$

Proof. Fix $x \in \mathbb{R}$. Let $\{a_n\} \subset \mathbb{R}$ be a sequence so that $0 < a_{n+1} < a_n$ for each n and $a_n \to 0$. Since g is defined as an integral with respect to y over $[0, 1]$, which has finite measure, let us restrict $\mathbb{R}^2$ to $A = [x, x + a_1] \times [0, 1]$, which is compact.
Recall, for any $y \in [0, 1]$, $\partial_x f(x, y) = \lim_{n\to\infty} \dfrac{f(x + a_n, y) - f(x, y)}{a_n}$. Also by mean value theorem, for each n there is $t_n \in (x, x + a_n)$ so that

$$h_n(y) = \partial_x f(t_n, y) = \frac{f(x + a_n, y) - f(x, y)}{a_n}.$$

Also since $\partial_x f$ is continuous, $\lim_{n\to\infty} h_n(y) = \partial_x f(x,y)$. Note this sequence $\{t_n\}$ may depend on y. So we have $h_n(y) \to \partial_x f(x,y)$ pointwise on $[0,1]$, which again has finite measure. Also since $\partial_x f$ is continuous on A, h_n is measurable on $[0,1]$ for each n.
Now remember that since we restricted down to A, which is compact, and since $\partial_x f$ is continuous on A, $\{h_n\}$ is uniformly bounded on $[0,1]$.
Therefore by definition of derivative, linearity of Lebesgue integral, and Bounded Convergence Theorem, we have

$$g'(x) = \lim_{n\to\infty} \frac{g(x+a_n) - g(x)}{a_n} = \lim_{n\to\infty} \int_0^1 h_n(y)dy = \int_0^1 \partial_x f(x,y)dy.$$

Also the continuity of $\partial_x f$ over $[0,1]$ with the fixed x, the integral on the right hand side exists and therefore $g'(x)$ exists. □

Problem RA 2: Let $\{f_k\}$ be a sequence of measurable functions in $\mathbb{R}^d$. Suppose that

$$g(x) = \lim_{k\to\infty} f_k(x)$$

exists for a.e. $x \in \mathbb{R}^d$. Show that g is measurable in $\mathbb{R}^d$.

Proof. Let $E \subset \mathbb{R}^d$ be the subset where g exists. Let $c \in \mathbb{R}$. Since the functions in the sequence are measurable, $\{x \in E : f_n(x) > c\}$ is measurable for any $n \in \mathbb{N}$.
Now let $k \in \mathbb{N}$. Let $h_k(x) = \sup_{n\geq k} f_n(x)$. Consider

$$\{x \in E : h_k(x) > c\} = \bigcup_{k=n}^{\infty} \{x \in E : f_n(x) > c\}.$$

Therefore $\{x \in E : h_k(x) > c\}$ is measurable. Since $c \in \mathbb{R}$ is arbitrary, $h_k(x)$ is measurable in E.
Now notice since g exists in E, $g(x) = \inf_{k\in\mathbb{N}} h_k(x)$ for every $x \in E$. Consider

$$\{x \in E : g(x) > c\} = \bigcap_{k=1}^{\infty} \{x \in E : h_k(x) > c\}.$$

Then $\{x \in E : g(x) > c\}$ is measurable. Since $c \in \mathbb{R}$ is arbitrary, g is measurable in $E \subset \mathbb{R}^d$. □

Problem RA 3: Suppose that f is a continuous real-valued function and consider the curve

$$\Gamma = \{(x, f(x)) : x \in \mathbb{R}\} \subset \mathbb{R}^2 .$$

Prove that Γ has measure zero in $\mathbb{R}^2$. Let μ denote Lebesgue measure in $\mathbb{R}^2$.

Proof. For each $j \in \mathbb{Z}$, let $I_j = [j, j+1]$. Note that $\mathbb{R} = \bigcup_{j\in\mathbb{Z}} I_j$. For each j, let $\Gamma_j = \{(x, f(x)) : x \in I_j\}$. Then $\Gamma = \bigcup_{j\in\mathbb{Z}} \Gamma_j$.
Fix $j \in \mathbb{Z}$. Since f is continuous and I_j is compact, f is uniformly continuous on I_j. Let $\varepsilon > 0$. Then there is $\delta > 0$ so that for any $x, y \in I_j$ so that $|x - y| < \delta$, $|f(x) - f(y)| < \varepsilon$. Note that Archimedes says there is a $n \in \mathbb{N}$, so that $\frac{1}{2^{n+|j|+1}} < \delta$.
Let $\{I_{j,k}\}_{k=1}^{2^{n+|j|+1}}$ be a collection of closed sub-intervals of I_j of length $\frac{1}{2^{n+|j|+1}}$. Note $\Gamma_j \subset \bigcup_{k=1}^{2^{n+|j|+1}} \Big(I_{j,k} \times f(I_{j,k})\Big)$. Then

$$\mu(\Gamma_j) \leq \sum_{k=1}^{2^{n+|j|+1}} \mu\Big(I_{j,k} \times f(I_{j,k})\Big) < \sum_{k=1}^{2^{n+|j|+1}} \frac{\varepsilon}{2^{n+|j|+1}} = \varepsilon .$$

Since we may take ε arbitrarily small, we get $\mu(\Gamma_j) = 0$.
Therefore

$$\mu(\Gamma) \leq \sum_{j\in\mathbb{Z}} \mu(\Gamma_j) = 0.$$

So $\mu(\Gamma) = 0$. □

Problem RA 4: Let $E \in \mathbb{R}^d$ be measurable and $0 < m(E) < \infty$. Prove that there exists a measurable set $F \subset E$ such that $m(F) = m(E)$, and

$$m(E \cap B_r(x)) > 0 \quad \forall x \in F \quad \textit{and} \quad \forall r > 0.$$

Proof. Let $Z = \{x \in E : m(E \cap B_r(x)) = 0 \quad \text{for some} \quad r > 0\}$.
Let $x \in Z$. Then there is $r_x > 0$ so that $m(E \cap B_{r_x}(x)) = 0$. Then for any $0 < c < r$, $B_c(x) \subset B_{r_x}(x)$ and hence $E \cap B_c(x) \subset E \cap B_{r_x}(x)$. Then $m(E \cap B_c(x)) \leq m(E \cap B_{r_x}(x)) = 0$. Therefore for any $n \in \mathbb{N}$,

$$\lim_{n\to\infty} \frac{m(E \cap B_{r_x/n}(x))}{B_{r_x/n}(x)} = 0.$$

Therefore x is not a point of Lebesgue density of E. Since E is measurable, a.e. $x \in E$ is a point of Lebesgue density. Therefore Z is contained in a subset of E that has measure zero. Therefore $m(Z) = 0$. This in turn gives us Z is measurable, and therefore $\mathbb{R}^d \setminus Z$ is measurable.
Let $F = E \setminus Z$. Note $F = E \cap (\mathbb{R}^d \setminus Z)$. Hence F is measurable. Also since E and Z are measurable, $m(F) = m(E) - m(Z) = m(E)$. □