Problem AC 1: Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and that

$$\lim_{n \to \infty} f(x) = \lim_{x \to -\infty} f(x) = 0.$$ 

Prove that $f$ is uniformly continuous.

Proof. Look to Problem AC 2 on January 2020 prelim solutions.

Problem AC 2: Suppose $\{I_k\}_{k \in \mathbb{N}}$ is a sequence of closed intervals $I_k = [a_k, b_k]$, such that $I_{k+1} \subseteq I_k$. Show that

$$C = \bigcap_{k=1}^{\infty} I_k$$

is non-empty. Must this be true if the $I_k$ are open?

Proof. First note that for any $i, j \in \mathbb{N}$, $a_i \leq b_j$. If $i \leq j$, then $a_i \leq a_j \leq b_j$. If $i > j$, then $a_i \leq b_i \leq b_j$.

Therefore $a = \sup \{a_k\}$ exists and $a \leq b_j$ for each $j$.

For each $n \in \mathbb{N}$, there is $a_{k_n}$ so that $a - \frac{1}{n} < a_{k_n} \leq a$. So $a_{k_n} \uparrow a$.

Let $k \in \mathbb{N}$. There is $N$ so that $k_n \geq k$ for each $n \geq N$. Therefore $a_k \leq a_{k_n} \leq a \leq b_{k_n} \leq b_k$ for each $n \geq N$. Since $I_k$ closed, $a, a_k, b_k \in I_k$.

Therefore $a \in I_k$ for each $k$. So $a \in C$, and thus $C$ is non-empty.

However if the $I_k$ are open, this is not necessarily true. For example, if for each $k \in \mathbb{N}$, $I_k = (1 - \frac{1}{k}, 1)$. Then $C$ empty.

Problem AC 3: Suppose $\{a_n\}$ is a sequence of positive numbers such that the limit

$$L = \lim_{n \to \infty} \frac{a_{n+1}}{a_n}$$

exists and is less than 1. Show directly that

$$\sum_{n=1}^{\infty} a_n$$

converges.

Proof. First note that since the sequence is positive, $L$ is nonnegative.

Second if we take $r \in (L, 1)$ then there is $N$ so that for $n > N$, $\frac{a_{n+1}}{a_n} < r$. Then for each
$k \in \mathbb{N}$, $a_{N+k} < r^ka_N$. Also, since $0 < r < 1$, $\sum_{k=1}^{\infty} a_N r^k < \infty$.

Therefore

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{k=1}^{\infty} a_{N+k} < \sum_{n=1}^{N} a_n + \sum_{k=1}^{\infty} a_N r^k < \infty.$$  

\[ \square \]

**Problem AC 4:** Consider the sequence defined recursively by $a_1 = 1$ and $a_{n+1} = \sqrt{2} a_n$ for $n \geq 1$. Show that this sequence converges, and determine the limit.

**Proof.** Claim: $a_n \leq a_{n+1} \leq 2$, for each $n$.

We will use induction to show $a_n \leq a_{n+1}$. We can see $a_1 = 1 \leq \sqrt{2} = a_2$. Suppose this is true for $n \geq 1$. Consider $a_{n+2} = \sqrt{2} a_{n+1} \geq \sqrt{2} a_n = a_{n+1}$.

Now we will use induction to show $a_n \leq 2$ for each $n$. Again we see $a_1 = 1 \leq 2$. Suppose this is true for $n \geq 1$. Then $a_{n+1} = \sqrt{2} a_n \leq \sqrt{4} = 2$.

With this claim, the sequence converges to some $a \in \mathbb{R}$ by monotone convergence theorem for sequences. Now what is the limit? Note

$$a = \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \sqrt{2} a_n = \sqrt{2} a.$$ 

If we solve for $a$ and recall that everything in the sequence and hence $a$ is greater than 0, we find $a = 2$. \[ \square \]

**Problem RA 1:** Let $f(x, y)$ be a continuous function in $\mathbb{R}^2$. Assume that $\partial_x f$ exists and is continuous in $\mathbb{R}^2$. Let

$$g(x) = \int_0^1 f(x, y) dy.$$ 

Show that $g$ is differentiable and

$$g'(x) = \int_0^1 \partial_x f(x, y) dy.$$ 

**Proof.** Fix $x \in \mathbb{R}$. Let $\{a_n\} \subset \mathbb{R}$ be a sequence so that $0 < a_{n+1} < a_n$ for each $n$ and $a_n \to 0$.

Since $g$ is defined as an integral with respect to $y$ over $[0, 1]$, which has finite measure, let us restrict $\mathbb{R}^2$ to $A = [x, x + a_1] \times [0, 1]$, which is compact.

Recall, for any $y \in [0, 1]$, $\partial_x f(x, y) = \lim_{n \to \infty} \frac{f(x + a_n, y) - f(x, y)}{a_n}$. Also by mean value theorem, for each $n$ there is $t_n \in (x, x + a_n)$ so that

$$h_n(y) = \partial_x f(t_n, y) = \frac{f(x + a_n, y) - f(x, y)}{a_n}.$$ 

Also since \( \partial_x f \) is continuous, \( \lim_{n \to \infty} h_n(y) = \partial_x f(x, y) \). Note this sequence \( \{t_n\} \) may depend on \( y \). So we have \( h_n(y) \to \partial_x f(x, y) \) pointwise on \([0, 1]\), which again has finite measure. Also since \( \partial_x f \) is continuous on \( A \), \( h_n \) is measurable on \([0, 1]\) for each \( n \).

Now remember that since we restricted down to \( A \), which is compact, and since \( \partial_x f \) is continuous on \( A \), \( \{h_n\} \) is uniformly bounded on \([0, 1]\).

Therefore by definition of derivative, linearity of Lebesgue integral, and Bounded Convergence Theorem, we have
\[
g'(x) = \lim_{n \to \infty} \frac{g(x + a_n) - g(x)}{a_n} = \lim_{n \to \infty} \int_0^1 h_n(y)dy = \int_0^1 \partial_x f(x, y)dy.
\]

Also the continuity of \( \partial_x f \) over \([0, 1]\) with the fixed \( x \), the integral on the right hand side exists and therefore \( g'(x) \) exists.

**Problem RA 2:** Let \( \{f_k\} \) be a sequence of measurable functions in \( \mathbb{R}^d \). Suppose that
\[
g(x) = \lim_{k \to \infty} f_k(x)
\]
exists for a.e. \( x \in \mathbb{R}^d \). Show that \( g \) is measurable in \( \mathbb{R}^d \).

**Proof.** Let \( E \subset \mathbb{R}^d \) be the subset where \( g \) exists. Let \( c \in \mathbb{R} \). Since the functions in the sequence are measurable, \( \{x \in E : f_n(x) > c \} \) is measurable for any \( n \in \mathbb{N} \).

Now let \( k \in \mathbb{N} \). Let \( h_k(x) = \sup_{n \geq k} f_n(x) \). Consider
\[
\{x \in E : h_k(x) > c\} = \bigcup_{k=n}^{\infty} \{x \in E : f_n(x) > c\}.
\]

Therefore \( \{x \in E : h_k(x) > c\} \) is measurable. Since \( c \in \mathbb{R} \) is arbitrary, \( h_k(x) \) is measurable in \( E \).

Now notice since \( g \) exists in \( E \), \( g(x) = \inf_{k \in \mathbb{N}} h_k(x) \) for every \( x \in E \). Consider
\[
\{x \in E : g(x) > c\} = \bigcap_{k=1}^{\infty} \{x \in E : h_k(x) > c\}.
\]

Then \( \{x \in E : g(x) > c\} \) is measurable. Since \( c \in \mathbb{R} \) is arbitrary, \( g \) is measurable in \( E \subset \mathbb{R}^d \).
**Problem RA 3:** Suppose that $f$ is a continuous real-valued function and consider the curve 

$$\Gamma = \{(x, f(x)) : x \in \mathbb{R}\} \subset \mathbb{R}^2.$$ 

Prove that $\Gamma$ has measure zero in $\mathbb{R}^2$. Let $\mu$ denote Lebesgue measure in $\mathbb{R}^2$.

*Proof.* For each $j \in \mathbb{Z}$, let $I_j = [j, j + 1]$. Note that $\mathbb{R} = \bigcup_{j \in \mathbb{Z}} I_j$. For each $j$, let $\Gamma_j = \{(x, f(x)) : x \in I_j\}$. Then $\Gamma = \bigcup_{j \in \mathbb{Z}} \Gamma_j$.

Fix $j \in \mathbb{Z}$. Since $f$ is continuous and $I_j$ is compact, $f$ is uniformly continuous on $I_j$. Let $\varepsilon > 0$. Then there is $\delta > 0$ so that for any $x, y \in I_j$ so that $|x - y| < \delta$, $|f(x) - f(y)| < \varepsilon$. Note that Archimedes says there is a $n \in \mathbb{N}$, so that $\frac{1}{2^{n+|j|+1}} < \delta$.

Let $\{I_{j,k}\}_{k=1}^{2^n+|j|+1}$ be a collection of closed sub-intervals of $I_j$ of length $\frac{1}{2^{n+|j|+1}}$. Note $\Gamma_j \subset \bigcup_{k=1}^{2^n+|j|+1} \left(I_{j,k} \times f(I_{j,k})\right)$. Then

$$\mu(\Gamma_j) \leq \sum_{k=1}^{2^n+|j|+1} \mu(I_{j,k} \times f(I_{j,k})) < \sum_{k=1}^{2^n+|j|+1} \frac{\varepsilon}{2^{n+|j|+1}} = \varepsilon.$$ 

Since we may take $\varepsilon$ arbitrarily small, we get $\mu(\Gamma_j) = 0$.

Therefore

$$\mu(\Gamma) \leq \sum_{j \in \mathbb{Z}} \mu(\Gamma_j) = 0.$$ 

So $\mu(\Gamma) = 0$. $\square$

**Problem RA 4:** Let $E \in \mathbb{R}^d$ be measurable and $0 < m(E) < \infty$. Prove that there exists a measurable set $F \subset E$ such that $m(F) = m(E)$, and

$$m(E \cap B_r(x)) > 0 \quad \forall x \in F \quad \text{and} \quad \forall r > 0.$$ 

*Proof.* Let $Z = \{x \in E : m(E \cap B_r(x)) = 0 \quad \text{for some} \quad r > 0\}$. Let $x \in Z$. Then there is $r_x > 0$ so that $m(E \cap B_{r_x}(x)) = 0$. Then for any $0 < c < r$, $B_c(x) \subset B_{r_x}(x)$ and hence $E \cap B_c(x) \subset E \cap B_{r_x}(x)$. Therefore for any $n \in \mathbb{N}$,

$$\lim_{n \to \infty} \frac{m(E \cap B_{r_x/n}(x))}{B_{r_x/n}(x)} = 0.$$ 

Therefore $x$ is not a point of Lebesgue density of $E$. Since $E$ is measurable, a.e. $x \in E$ is a point of Lebesgue density. Therefore $Z$ is contained in a subset of $E$ that has measure zero. Therefore $m(Z) = 0$. This in turn gives us $Z$ is measurable, and therefore $\mathbb{R}^d \setminus Z$ is measurable.

Let $F = E \setminus Z$. Note $F = E \cap (\mathbb{R}^d \setminus Z)$. Hence $F$ is measurable. Also since $E$ and $Z$ are measurable, $m(F) = m(E) - m(Z) = m(E)$. $\square$