**Problem AC1:** Suppose that \( \lim_{x \to 0} f(x) = \infty \) and \( \lim_{x \to 0} g(x) = L \) for some \( L \in \mathbb{R} \). Show that \( \lim_{x \to 0} (f(x) + g(x)) = \infty \).

*Proof.* We are told \( \lim_{x \to 0} g(x) = L \). Therefore there is a \( \delta_1 > 0 \) such that when \( |x| < \delta_1 \), \( |g(x) - L| < 1 \). We can manipulate this to say \( g(x) > L - 1 \) when \( |x| < \delta_1 \).
Let \( M > 0 \). We are told \( \lim_{x \to 0} f(x) = \infty \). Therefore there is \( \delta_2 > 0 \) so that when \( |x| < \delta_2 \), \( f(x) > M - L + 1 \).
Let \( \delta = \min\{\delta_1, \delta_2\} \) and suppose \( |x| < \delta \). Then \( f(x) + g(x) > M - L + 1 + L - 1 = M \).
Since \( M \) was an arbitrary positive number, we have the desired result. \( \square \)

**Problem AC2:** Suppose that \( f \) is continuous on \([0, \infty)\) and that \( \lim_{x \to \infty} f(x) = 0 \). Show that \( f \) is uniformly continuous on \([0, \infty)\).

*Proof.* Let \( \varepsilon > 0 \). Then since \( \lim_{x \to \infty} f(x) = 0 \), there exists an \( M > 0 \) so that when \( x \geq M \), \( |f(x)| < \frac{\varepsilon}{3} \).
Consider \( f|_{[0,M]} \). Since \([0, M]\) compact and \( f \) is continuous, \( f \) is uniformly continuous on \([0, M]\). Therefore there is a \( \delta > 0 \) so that for any \( x, y \in [0, M] \) with \( |x - y| < \delta \), \( |f(x) - f(y)| < \frac{\varepsilon}{3} < \varepsilon \). By above, for any \( x, y \in [M, \infty) \), \( |f(x) - f(y)| \leq |f(x)| + |f(y)| < \frac{2\varepsilon}{3} < \varepsilon \). So this will definitely hold when \( x, y \in [M, \infty) \) are so that \( |x - y| < \delta \).
Now suppose \( 0 < x \leq M < y \) and \( |x - y| < \delta \). Then \( |x - M| < \delta \). Therefore \( |f(x) - f(y)| = |f(x) - f(M) + f(M) - f(y)| \leq |f(x) - f(M)| + |f(M) - f(y)| \leq |f(x) - f(M)| + |f(M)| + |f(y)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \).
Therefore \( f \) is uniformly continuous on \([0, \infty)\). \( \square \)

**Problem AC3:** Suppose \( f : [a, b] \to \mathbb{R} \) is bounded and nondecreasing.

(a) Show that the limits \( f(x^+) = \lim_{y \to x, y > x} f(y) \) exists for every \( x \in [a, b] \) and the limits \( f(x^-) = \lim_{y \to x, y < x} f(y) \) for every \( x \in (a, b) \).

(b) Show that \( f \) has at most countably many jump discontinuities.
Proof. (a) Since $f$ is bounded, there is $M > 0$ so that $|f(x)| < M$ for all $x \in [a,b]$. Let $x \in [a,b)$. Consider the sequence $y_n = x + \frac{b-x}{2^n}$. Note that $y_n > y_{n+1} > x$ for every $n$ and $y_n \to x$. Therefore for each $n$, $-M < f(x) \leq f(y_n)$ and $f(y_n) \geq f(y_{n+1})$. Therefore by monotone convergence theorem of sequences, there exists $L \in \mathbb{R}$ so that $L = \lim_{n \to \infty} f(y_n) = \lim_{y \to x, y > x} f(y)$. Denote this $L = f(x^+)$. A very similar argument can be made for the other limits.

(b) Recall that the rational numbers are dense in $\mathbb{R}$, and they are a countable subset of $\mathbb{R}$. Let $S = \{x \in [a, b] : f \text{ has a jump discontinuity}\}$. Then for each $x \in S$, by part (a) there exists $f(x^-)$ and $f(x^+)$, and since $f$ is nondecreasing $f(x^-) < f(x^+)$. By density of the rationals, there is a rational number $q_x$ so that $f(x^-) < q_x < f(x^+)$. Consider the map $S \to \mathbb{Q}$ via $x \mapsto q_x$. I claim this map is injective: Indeed suppose $x, t \in S$ and $x \neq t$. Then WLOG $x < t$. Since $f$ is nondecreasing $q_x < f(x^+) \leq f(t^-) < q_t$. So $q_x \neq q_t$ and hence this map is injective. Therefore $S$ is bijective with a subset of the rationals, which is at most countable. Therefore $S$ is at most countable.

Problem AC4: Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable, and $f(0) = f'(0) = 0$. Show that if \{a_n\} is nonnegative and

$$
\sum_{n=1}^{\infty} a_n < \infty
$$

then

$$
\sum_{n=1}^{\infty} f(a_n) < \infty.
$$

Proof. First, since $\sum_{n=1}^{\infty} a_n < \infty$, $a_n \to 0$ as $n \to \infty$. Also since $f$ is differentiable and hence continuous, $f(a_n) \to f(0) = 0$ as $n \to \infty$.

Second, since $f(0) = f'(0) = 0$, $\lim_{n \to \infty} \left| \frac{f(a_n)}{a_n} \right| = 0$. Therefore there exists $N \in \mathbb{N}$ so that $\left| \frac{f(a_n)}{a_n} \right| < 1$ for $n \geq N$.

With these, we have

$$
\sum_{n=1}^{\infty} |f(a_n)| = \sum_{n=1}^{N-1} |f(a_n)| + \sum_{n=N}^{\infty} |f(a_n)| < \sum_{n=1}^{N-1} |f(a_n)| + \sum_{n=N}^{\infty} a_n < \infty.
$$
Problem RA1: Suppose that \( f \) is a finite-valued measurable function on \([0, 1]\) and that \( |f(x) - f(y)| \) is integrable on \( A = [0, 1] \times [0, 1] \). Show that \( f \) is integrable on \([0, 1]\).

Proof. By Fubini (or Tonelli, one of them), since \( |f(x) - f(y)| \) is integrable over \( A \),
\[
\infty > \int_A |f(x) - f(y)| dm = \int_0^1 \left( \int_0^1 |f(x) - f(y)| dx \right) dy.
\]

Therefore \( \int_0^1 |f(x) - f(y)| dx < \infty \) for almost every \( y \in [0, 1] \). Let \( y_0 \) be one of those nice elements in \([0, 1]\). Since \( f \) is finite valued, \( |f(y_0)| < \infty \). Consider
\[
\int_0^1 |f(x)| dx \leq \int_0^1 |f(x) - f(y_0)| dx + \int_0^1 |f(y_0)| dx < \infty.
\]
Therefore \( f \) is integrable over \([0, 1]\). \( \square \)

Problem RA2: Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is absolutely continuous. Show:
(a) \( f \) maps sets of measure zero to sets of measure zero.
(b) \( f \) maps measurable sets to measurable sets.

Proof. (a) Let \( \varepsilon > 0 \). Since \( f \) is absolutely continuous, there is a \( \delta > 0 \) so that for any countable collection of intervals \( \{I_k = [a_k, b_k]\}_{k=1}^\infty \) so that \( \sum_{k=1}^\infty |I_k| < \delta \), \( \sum_{k=1}^\infty |f(b_k) - f(a_k)| < \varepsilon \). Note that since \( f \) is continuous, we may break up the intervals at points of relative extrema and thus making \( |f(b_k) - f(a_k)| = |f(I_k)| \) for each \( k \).

Let \( A \) be a set of measure zero. Then for that \( \delta \) we mentioned before, there is a countable collection of closed bounded intervals \( \{I_k\} \) so that \( A \subset \bigcup_{k=1}^\infty I_k \) and \( \sum_{k=1}^\infty |I_k| < \delta \). Therefore by absolute continuity, \( m(f(A)) \leq m(\bigcup_{k=1}^\infty f(I_k)) \leq \sum_{k=1}^\infty |f(I_k)| < \varepsilon \). Hence \( f(A) \) has measure zero.

(b) Let \( A \in \mathbb{R} \) be measurable. Define for each positive integer \( n \), \( E_n = [-n, n] \). Note that each \( E_n \) is measurable, and therefore \( A \cap E_n \) is measurable for each \( n \). Also, \( A = A \cap \left( \bigcup_{n=1}^\infty E_n \right) = \bigcup_{n=1}^\infty (A \cap E_n) \).

Fix a positive integer \( n \): Since \( A \cap E_n \) is measurable, there is a closed set \( F_n \subset A \cap E_n \) and a set \( Z_n = (A \cap E_n) \setminus F_n \) of measure zero so that \( A \cap E_n = Z_n \cup F_n \). Note that by construction \( F_n \) is compact. Then \( f(F_n) \) is compact. Also by part (a), \( f(Z_n) \) has measure zero. Also \( f(A \cap E_n) = f(Z_n) \cup f(F_n) \). Then since \( f(A \cap E_n) \) differs from a compact set by a set of measure zero, \( f(A \cap E_n) \) is measurable.
Therefore \( f(A) = f \left( \bigcup_{n=1}^\infty A \cap E_n \right) = \bigcup_{n=1}^\infty f(A \cap E_n) \) is measurable, as desired. \( \square \)
**Problem RA3:** Suppose that $f$ is integrable on $\mathbb{R}^d$ and

$$E_\alpha = \{ x : |f(x)| > \alpha \}.$$  

Prove that

$$\int_{\mathbb{R}^d} |f(x)| dx = \int_0^\infty m(E_\alpha) d\alpha.$$  

**Proof.** The trick is to consider the function $g : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ via $g(\alpha, x) = \chi_{(0, \infty)}(\alpha) \chi_{E_\alpha}(x)$. Notice that $(0, \infty)$ is measurable, and therefore $\chi_{(0, \infty)}(\alpha)$ is measurable. Then by corollary 3.7 on page 85, $g_1(\alpha, x) = \chi_{(0, \infty)}(\alpha)$ is measurable over $\mathbb{R} \times \mathbb{R}^d$. Also $f$ is integrable. So $E_\alpha$ is measurable, and therefore $\chi_{E_\alpha}(x)$ is measurable on $\mathbb{R}^d$. Then $g_2(\alpha, x) = \chi_{E_\alpha}(x)$ is measurable over $\mathbb{R} \times \mathbb{R}^d$. Therefore $g(\alpha, x) = g_1(\alpha, x) g_2(\alpha, x)$ is measurable over $\mathbb{R} \times \mathbb{R}^d$. Since $g$ is a nonnegative measurable function, by Tonelli, we have

$$\int_\mathbb{R} \left( \int_{\mathbb{R}^d} \chi_{(0, \infty)}(\alpha) \chi_{E_\alpha}(x) dx \right) d\alpha = \int_{\mathbb{R}^d} \left( \int_\mathbb{R} \chi_{(0, \infty)}(\alpha) \chi_{E_\alpha}(x) d\alpha \right) dx. \tag{1}$$

Note $\chi_{(0, \infty)}(\alpha)$ is zero for $\alpha \leq 0$. Therefore we may restrict $\mathbb{R}$ to $(0, \infty)$. Let $\alpha > 0$. Then

$$\int_{\mathbb{R}^d} \chi_{(0, \infty)}(\alpha) \chi_{E_\alpha}(x) dx = \int_{\mathbb{R}^d} \chi_{E_\alpha}(x) dx = m(E_\alpha).$$

Fix $x \in \mathbb{R}^d$. Then

$$\int_{\mathbb{R}^d} \chi_{(0, \infty)}(\alpha) \chi_{E_\alpha}(x) d\alpha = \int_0^\infty \chi_{E_\alpha}(x) d\alpha = \int_0^{\|f(x)\|} d\alpha = |f(x)|.$$  

If we plug these results into equation (1), we get

$$\int_0^\infty m(E_\alpha) d\alpha = \int_{\mathbb{R}^d} |f(x)| dx.$$  

Special thanks to those who helped me come to this solution.

**Problem RA4:** Suppose $\{f_n\}$ is a sequence of integrable functions on $\mathbb{R}$ such that

$$\sum_{n=1}^\infty \int_{\mathbb{R}} |f_{n+1} - f_n| < \infty.$$  

Show that there is a function $f$ such that

$$\lim_{n \to \infty} \int_{\mathbb{R}} |f_n - f| = 0.$$
Proof. We can answer this problem from the perspective of normed spaces. Recall that \( L^1(\mathbb{R}) \) is a complete normed linear space under the norm \( \|f\| = \int_{\mathbb{R}} |f| \).

The problem gives us a sequence \( \{f_n\} \in L^1(\mathbb{R}) \) with the property that \( \sum_{n=1}^{\infty} \|f_{n+1} - f_n\| < \infty \).

Let \( \varepsilon > 0 \). By this property, there exists an \( N \) so that
\[
\sum_{n=N}^{\infty} \|f_{n+1} - f_n\| < \varepsilon.
\]

Let \( m > k \geq N \). Then
\[
\|f_m - f_k\| \leq \sum_{n=k}^{m-1} \|f_{n+1} - f_n\| \leq \sum_{n=k}^{\infty} \|f_{n+1} - f_n\| < \varepsilon.
\]

Therefore our sequence is Cauchy in \( L^1(\mathbb{R}) \). Since \( L^1(\mathbb{R}) \) is complete, there exists a function \( f \in L^1(\mathbb{R}) \) so that \( \lim_{n \to \infty} \|f_n - f\| = 0 \).

In other words
\[
\lim_{n \to \infty} \int_{\mathbb{R}} |f_n - f| = 0.
\]
\(\square\)