Problem AC1: Suppose that $\lim_{x\to 0} f(x) = \infty$ and $\lim_{x\to 0} g(x) = L$ for some $L \in \mathbb{R}$. Show that $\lim_{x\to 0}(f(x) + g(x)) = \infty$.

Proof. We are told $\lim_{x\to 0} g(x) = L$. Therefore there is a $\delta_1 > 0$ such that when $|x| < \delta_1$, $|g(x) - L| < 1$. We can manipulate this to say $g(x) > L - 1$ when $|x| < \delta_1$.
Let $M > 0$. We are told $\lim_{x\to 0} f(x) = \infty$. Therefore there is $\delta_2 > 0$ so that when $|x| < \delta_2$, $f(x) > M - L + 1$.
Let $\delta = min\{\delta_1, \delta_2\}$ and suppose $|x| < \delta$. Then $f(x) + g(x) > M - L + 1 + L - 1 = M$. Since M was an arbitrary positive number, we have the desired result. □

Problem AC2: Suppose that f is continuous on $[0, \infty)$ and that $lim_{x\to\infty} f(x) = 0$. Show that f is uniformly continuous on $[0, \infty)$.

Proof. Let $\varepsilon > 0$. Then since $\lim_{x\to\infty} f(x) = 0$, There exists an $M > 0$ so that when $x \geq M$, $|f(x)| < \frac{\varepsilon}{3}$.
Consider $f\big|_{[0,M]}$. Since $[0, M]$ compact and f is continuous, f is uniformly continuous on $[0, M]$. Therefore there is a $\delta > 0$ so that for any $x, y \in [0, M]$ with $|x-y| < \delta$, $|f(x)-f(y)| < \frac{\varepsilon}{3} < \varepsilon$.
By above, for any $x, y \in [M, \infty)$, $|f(x) - f(y)| \leq |f(x)| + |f(y)| < \frac{2\varepsilon}{3} < \varepsilon$. So this will definitely hold when $x, y \in [M, \infty)$ are so that $|x - y| < \delta$.
Now suppose $0 < x \leq M < y$ and $|x - y| < \delta$. Then $|x - M| < \delta$. Therefore

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f(M) + f(M) - f(y)| \leq |f(x) - f(M)| + |f(M) - f(y)| \\ &\leq |f(x) - f(M)| + |f(M)| + |f(y)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

Therefore f is uniformly continuous on $[0, \infty)$. □

Problem AC3: Suppose $f : [a, b] \to \mathbb{R}$ is bounded and nondecreasing.
(a) Show that the limits

$$f(x^+) = \lim_{y\to x, y>x} f(y)$$

exists for every $x \in [a, b)$ and the limits

$$f(x^-) = \lim_{y\to x, y<x} f(y)$$

for every $x \in (a, b]$.

(b) Show that f has at most countably many jump discontinuities.

Proof. (a) Since f is bounded, there is $M > 0$ so that $|f(x)| < M$ for all $x \in [a, b]$. Let $x \in [a, b)$. Consider the sequence $y_n = x + \frac{b-x}{2n}$. Note that $y_n > y_{n+1} > x$ for every n and $y_n \to x$. Therefore for each n, $-M < f(x) \leq f(y_n)$ and $f(y_n) \geq f(y_{n+1})$. Therefore by monotone convergence theorem of sequences, there exists $L \in \mathbb{R}$ so that $L = \lim_{n\to\infty} f(y_n) = \lim_{y\to x, y>x} f(y)$. Denote this $L = f(x^+)$.
A very similar argument can be made for the other limits.

(b) Recall that the rational numbers are dense in $\mathbb{R}$, and they are a countable subset of $\mathbb{R}$. Let

$$S = \{x \in [a, b] : f \text{ has a jump discontinuity}\}.$$

Then for each $x \in S$, by part *(a)* there exists $f(x^-)$ and $f(x^+)$, and since f is nondecreasing $f(x^-) < f(x^+)$. By density of the rationals, there is a rational number q_x so that $f(x^-) < q_x < f(x^+)$.
Consider the map $S \to \mathbb{Q}$ via $x \mapsto q_x$. I claim this map is injective: Indeed suppose $x, t \in S$ and $x \neq t$. Then WLOG $x < t$. Since f is nondecreasing $q_x < f(x^+) \leq f(t^-) < q_t$. So $q_x \neq q_t$ and hence this map is injective.
Therefore S is bijectiive with a subset of the rationals, which is at most countable. Therefore S is at most countable. □

Problem AC4: Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable, and $f(0) = f'(0) = 0$. Show that if $\{a_n\}$ is nonnegative and

$$\sum_{n=1}^{\infty} a_n < \infty$$

then

$$\sum_{n=1}^{\infty} f(a_n) < \infty.$$

Proof. First, since $\sum_{n=1}^{\infty} a_n < \infty$, $a_n \to 0$ as $n \to \infty$. Also since f is differentiable and hence continuous, $f(a_n) \to f(0) = 0$ as $n \to \infty$.
Second, since $f(0) = f'(0) = 0$, $\lim_{n\to\infty} |\frac{f(a_n)}{a_n}| = 0$. Therefore there exists $N \in \mathbb{N}$ so that $\frac{|f(a_n)|}{a_n} < 1$ for $n \geq N$.
With these, we have

$$\sum_{n=1}^{\infty} |f(a_n)| = \sum_{n=1}^{N-1} |f(a_n)| + \sum_{n=N}^{\infty} |f(a_n)| < \sum_{n=1}^{N-1} |f(a_n)| + \sum_{n=N}^{\infty} a_n < \infty.$$

□

Problem RA1: Suppose that f is a finite-valued measurable function on $[0,1]$ and that $|f(x)-f(y)|$ is integrable on $A=[0,1]\times[0,1]$. Show that f is integrable on $[0,1]$.

Proof. By Fubini (or Tonelli, one of them), since $|f(x)-f(y)|$ is integrable over A,

$$\infty > \int_A |f(x)-f(y)|dm = \int_0^1 \Big(\int_0^1 |f(x)-f(y)|dx \Big) dy.$$

Therefore $\int_0^1 |f(x)-f(y)|dx < \infty$ for almost every $y \in [0,1]$. Let y_0 be one of those nice elements in $[0,1]$. Since f is finite valued, $|f(y_0)| < \infty$. Consider

$$\int_0^1 |f(x)|dx \le \int_0^1 |f(x)-f(y_0)|dx + \int_0^1 |f(y_0)|dx < \infty.$$

Therefore f is integrable over $[0,1]$. □

Problem RA2: Suppose that $f:\mathbb{R}\to\mathbb{R}$ is absolutely continuous. Show:
(a) f maps sets of measure zero to sets of measure zero.
(b) f maps measurable sets to measurable sets.

Proof. (a) Let $\varepsilon > 0$. Since f is absolutely continuous, there is a $\delta > 0$ so that for any countable collection of intervals $\{I_k = [a_k, b_k]\}_{k=1}^\infty$ so that $\sum_{k=1}^\infty |I_k| < \delta$, $\sum_{k=1}^\infty |f(b_k) - f(a_k)| < \varepsilon$. Note that since f is continuous, we may break up the intervals at points of relative extrema and thus making $|f(b_k)-f(a_k)| = |f(I_k)|$ for each k.
Let A be a set of measure zero. Then for that δ we mentioned before, there is a countable collection of closed bounded intervals $\{I_k\}$ so that $A \subset \bigcup_{k=1}^\infty I_k$ and $\sum_{k=1}^\infty |I_k| < \delta$. Therefore by absolute continuity, $m(f(A)) \le m(\bigcup_{k=1}^\infty f(I_k)) \le \sum_{k=1}^\infty |f(I_k)| < \varepsilon$ Hence $f(A)$ has measure zero.

(b) Let $A \in \mathbb{R}$ be measurable. Define for each positive integer n, $E_n = [-n, n]$. Note that each E_n is measurable, and therefore $A \cap E_n$ is measurable for each n. Also, $A = A \cap \Big(\bigcup_{n=1}^\infty E_n \Big) = \bigcup_{n=1}^\infty (A \cap E_n)$.
Fix a positive integer n: Since $A\cap E_n$ is measurable, there is a closed set $F_n \subset A \cap E_n$ and a set $Z_n = (A\cap E_n)\setminus F_n$ of measure zero so that $A \cap E_n = Z_n \sqcup F_n$. Note that by construction F_n is compact. Then $f(F_n)$ is compact. Also by part *(a)*, $f(Z_n)$ has measure zero. Also $f(A\cap E_n) = f(Z_n) \cup f(F_n)$. Then since $f(A \cap E_n)$ differs from a compact set by a set of measure zero, $f(A\cap E_n)$ is measurable.
Therefore $f(A) = f\Big(\bigcup_{n=1}^\infty A \cap E_n\Big) = \bigcup_{n=1}^\infty f(A\cap E_n)$ is measurable, as desired. □

Problem RA3: Suppose that f is integrable on $\mathbb{R}^d$ and

$$E_\alpha = \{x : |f(x)| > \alpha\}.$$

Prove that

$$\int_{\mathbb{R}^d} |f(x)| dx = \int_0^\infty m(E_\alpha) d\alpha.$$

Proof. The trick is to consider the function $g : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ via $g(\alpha, x) = \chi_{(0,\infty)}(\alpha)\chi_{E_\alpha}(x)$. Notice that $(0, \infty)$ is measurable, and therefore $\chi_{(0,\infty)}(\alpha)$ is measurable. Then by corollary 3.7 on page 85, $\tilde{g_1}(\alpha, x) = \chi_{(0,\infty)}(\alpha)$ is measurable over $\mathbb{R} \times \mathbb{R}^d$. Also f is integrable. So E_α is measurable, and therefore $\chi_{E_\alpha}(x)$ is measurable on $\mathbb{R}^d$. Then $\tilde{g_2}(\alpha, x) = \chi_{E_\alpha}(x)$ is measurable over $\mathbb{R} \times \mathbb{R}^d$. Therefore $g(\alpha, x) = \tilde{g_1}(\alpha, x)\tilde{g_2}(\alpha, x)$ is measurable over $\mathbb{R} \times \mathbb{R}^d$. Since g is a nonnegative measurable function, by Tonelli, we have

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}^d} \chi_{(0,\infty)}(\alpha)\chi_{E_\alpha}(x) dx \right) d\alpha = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}} \chi_{(0,\infty)}(\alpha)\chi_{E_\alpha}(x) d\alpha \right) dx. \tag{1}$$

Note $\chi_{(0,\infty)}(\alpha)$ is zero for $\alpha \leq 0$. Therefore we may restrict $\mathbb{R}$ to $(0, \infty)$.
Let $\alpha > 0$. Then

$$\int_{\mathbb{R}^d} \chi_{(0,\infty)}(\alpha)\chi_{E_\alpha}(x) dx = \int_{\mathbb{R}^d} \chi_{E_\alpha}(x) dx = m(E_\alpha).$$

Fix $x \in \mathbb{R}^d$. Then

$$\int_{\mathbb{R}} \chi_{(0,\infty)}(\alpha)\chi_{E_\alpha}(x) d\alpha = \int_0^\infty \chi_{E_\alpha}(x) d\alpha = \int_0^{|f(x)|} d\alpha = |f(x)|.$$

If we plug these results into equation (1), we get

$$\int_0^\infty m(E_\alpha) d\alpha = \int_{\mathbb{R}^d} |f(x)| dx.$$

□

Special thanks to those who helped me come to this solution.

Problem RA4: Suppose $\{f_n\}$ is a sequence of integrable functions on $\mathbb{R}$ such that

$$\sum_{n=1}^\infty \int_{\mathbb{R}} |f_{n+1} - f_n| < \infty.$$

Show that there is a function f such that

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f_n - f| = 0.$$

Proof. We can answer this problem from the perspective of normed spaces. Recall that $L^1(\mathbb{R})$ is a complete normed linear space under the norm $\|f\| = \int_{\mathbb{R}} |f|$.
The problem gives us a sequence $\{f_n\} \in L^1(\mathbb{R})$ with the property that $\sum_{n=1}^{\infty} \|f_{n+1} - f_n\| < \infty$.
Let $\varepsilon > 0$. By this property, there exists an N so that

$$\sum_{n=N}^{\infty} \|f_{n+1} - f_n\| < \varepsilon.$$

Let $m > k \geq N$. Then $\|f_m - f_k\| \leq \sum_{n=k}^{m-1} \|f_{n+1} - f_n\| \leq \sum_{n=k}^{\infty} \|f_{n+1} - f_n\| < \varepsilon$.
Therefore our sequence is Cauchy in $L^1(\mathbb{R})$. Since $L^1(\mathbb{R})$ is complete, there exists a function $f \in L^1(\mathbb{R})$ so that $\lim_{n\to\infty} \|f_n - f\| = 0$.
In other words

$$\lim_{n\to\infty} \int_{\mathbb{R}} |f_n - f| = 0.$$

□