Problem AC1: Suppose $\{a_n\}$ and $\{b_n\}$ are sequences, and $\lim_{n\to\infty} a_n b_n = 0$. Show that at least one of $\{a_n\}$ or $\{b_n\}$ has a subsequence that converges to zero.

Proof. Suppose for contradiction that neither $\{a_n\}$ nor $\{b_n\}$ have a subsequence that converges to 0. Therefore there is a $\varepsilon > 0$ so that all but finitely many $a_n \geq \varepsilon$ and all but finitely many $b_n \geq \varepsilon$. Let N_1 be the largest nonnegative integer so that $a_{N_1} < \varepsilon$, and let N_2 be the largest nonnegative integer so that $b_{N_2} < \varepsilon$.
Let $N = max\{N_1 + 1, N_2 + 1\}$. Then for any $n \geq N$, $a_n, b_n \geq \varepsilon$.
Therefore with this $\varepsilon > 0$, for any n so that $1 \leq n < N$, there is N so that $a_N b_N > \varepsilon^2$. Also for any $n \geq N$, $a_n b_n > \varepsilon^2$. Thus $a_n b_n \not\to 0$. This is our contradiction. Therefore one or both of the sequences must have a subsequence that converges to 0. □

Problem AC2: Let $\{a_n\}$ be a sequence which converges to 0. Suppose $f : [0, 1] \to \mathbb{R}$ is a bounded function that is continuous except on $\{a_n\}$. Show that f is Riemann integrable on $[0, 1]$.

Proof. Since f is bounded, there is $M > 0$ so that $|f(x)| < M$ for all x.
Let $0 < \varepsilon < 1$, then there is $N_1 < N_2$ so that $|a_n| < 1$ for $n \geq N_1$ and $|a_n| < \varepsilon/4M$ for $n \geq N_2$. Consider the partition $P = \Big\{x_0, x_0 = |a_{N_2}|, x_1 = |a_{N_2-1}|, ..., x_{N_2-N_1} = |a_{N_1}|, 1\Big\}$. Then since f is continuous over $[x_{k-1}, x_k]$ for $k = 1, ..., N_2 - N_1 + 1$, it is Riemann integrable over each interval. Then for each $k = 1, ..., N_2 - N_1 + 1$ there are partitions P_k on $[x_{k-1}, x_k]$ so that $U(f, P_k) - L(f, P_k) < \dfrac{\varepsilon}{2(N_2 - N_1 + 1)}$.
Let $Q = P \cup \bigcup_{n=1}^{N_2-N_1+1} \{P_n\}$. Then

$$U(f, Q) - L(f, Q) = \sup_{s,t\in\left[0,|a_{N_2}|\right]} \Big(f(t) - f(s)\Big)|a_{N_2}| + \sum_{k=1}^{N_2-N_1+1} U(f, P_k) - L(f, P_k)$$
$$< 2M\frac{\varepsilon}{4M} + (N_2 - N_1 + 1)\frac{\varepsilon}{2(N_2 - N_1 + 1)} = \varepsilon\,.$$

If $\varepsilon \geq 1$, then there is an integer $W \geq 2$ so that $\varepsilon/W < 1$. Then repeat the argument above with ε/W.
Hence f is Riemann integrable over $[0, 1]$. □

Problem AC3: Let f be positive and continuous on $[0,\infty)$ and differentiable on $(0,\infty)$. Suppose $|f'(x)| < 1$ for all $x \in (0,\infty)$ and $\int_0^\infty f(x)dx < \infty$. Show that f is bounded on $[0,\infty)$.

Proof. Suppose for contradiction that f is not bounded. Then for each $n \in \mathbb{N}$, there is $x_n \in [0,\infty)$ so that $f(x_n) > n$.
Pick any $x, y \in [0,\infty)$. Then by the mean value theorem, there is $\xi_{x,y} \in (x,y)$ so that

$$\frac{|f(y) - f(x)|}{|y - x|} = |f'(\xi_{x,y})| < 1.$$

Therefore we have

$$|f(x) - f(y)| < |x - y| \quad \forall x, y \in [0,\infty).$$

Therefore f is uniformly continuous.
Then there is a $\delta > 0$ so that $|f(x_n) - f(x)| < 1/2$ when $|x_n - x| < \delta$ for any n. So we have

$$\int_0^\infty f(x)dx \geq \int_{x_n - \frac{\delta}{2}}^{x_n + \frac{\delta}{2}} f(x)dx > (f(n) - 1/2)\delta > (n - 1/2)\delta, \quad \forall n.$$

Let $n \to \infty$ and we see $\int_0^\infty f(x)dx = \infty$, which is a contradiction.
Thus f must be bounded. □

Problem AC4: Suppose $\{a_n\}$ is a positive sequence, and $\sum_{n=1}^\infty a_n < \infty$. Show that if g is continuously differentiable and $g(0) = 0$, then $\sum_{n=1}^\infty g(a_n) < \infty$.

Proof. Since $\sum_{n=1}^\infty a_n < \infty$, $a_n \to 0$. Also since g is continuously differentiable (hence continuous), $g(a_n) \to g(0) = 0$.
By the mean value theorem, for each n there is $\xi_n \in (0, a_n)$ so that $g(a_n) = g'(\xi_n)a_n$. Also since $\xi_n \to 0$, $g'(\xi_n) \to g'(0) < \infty$. Then there is an N so that $|g'(\xi_n) - g'(0)| < 1$ when $n \geq N$. Therefore we have,

$$\begin{aligned}
\sum_{n=1}^\infty g(a_n) = \sum_{n=1}^\infty g'(\xi_n)a_n &= g'(0)\sum_{n=1}^\infty a_n + \sum_{n=1}^\infty \big(g'(\xi_n) - g'(0)\big)a_n \\
&\leq g'(0)\sum_{n=1}^\infty a_n + \sum_{n=1}^{N-1} \big(g'(\xi_n) - g'(0)\big)a_n + \sum_{n=N}^\infty |g'(\xi_n) - g'(0)|a_n \\
&< g'(0)\sum_{n=1}^\infty a_n + \sum_{n=1}^{N-1} \big(g'(\xi_n) - g'(0)\big)a_n + \sum_{n=N}^\infty a_n < \infty.
\end{aligned}$$

□

Problem RA1: Let $f \in L^1(\mathbb{R})$ (i.e. f is integrable over $\mathbb{R}$), and define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = e^{-nx^2} f(x), \quad n \geq 1.$$

Show that $f_n \in L^1(\mathbb{R})$, and find $\lim_{n\to\infty} \int_{\mathbb{R}} f_n$.

Proof. Note that for $n \geq 1$ and for all $x \in \mathbb{R}$, $0 < e^{-nx^2} \leq 1$. Therefore for each n $|f_n(x)| = |f(x)|e^{-nx^2} \leq |f(x)|$. Since f is integrable, $|f(x)|$ is integrable. Then

$$\int_{\mathbb{R}} |f_n| \leq \int_{\mathbb{R}} |f| < \infty.$$

So $f_n \in L^1(\mathbb{R})$ for each n.
Now notice that $f_n(x) \to 0$ pointwise a.e. as $n \to \infty$. Remark: we can only say almost everywhere convergence to 0 since we only know $|f(x)|$ is finite almost everywhere. Also $|f_n| \leq |f|$ (which is integrable) for each n. Therefore by the LDCT, $\left\{ \int_{\mathbb{R}} f_n \right\} \to 0$. □

Problem RA2: Show that if $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous and Lesbesgue integrable, then

$$\lim_{|x|\to\infty} f(x) = 0.$$

Proof. Suppose for contradiction that $f(x) \not\to 0$ as $|x| \to \infty$. Let us first restrict to $[0, \infty)$. Then there is a $\varepsilon > 0$ so that for any n there is an $x_n \geq n$ so that $|f(x_n)| \geq \varepsilon$. Since f is uniformly continuous, there is $\delta > 0$ so that $|f(x) - f(x_n)| < \frac{\varepsilon}{2}$ whenever $|x - x_n| < \delta$, for any n.
Therefore

$$\int_{\mathbb{R}} |f(x)| \geq \sum_{n=1}^{\infty} \int_{x_n - \frac{\delta}{2}}^{x_n + \frac{\delta}{2}} |f(x)| > \sum_{n=1}^{\infty} \int_{x_n - \frac{\delta}{2}}^{x_n + \frac{\delta}{2}} |f(x_n) - \frac{\varepsilon}{2}| \geq \sum_{n=1}^{\infty} \frac{\varepsilon}{2}\delta = \infty.$$

This contradicts f being integrable.
We can do the same argument on $(-\infty, 0]$. So $f(x) \to 0$ as $|x| \to \infty$. □

Problem RA 3: Show that a function, f, that is absolutely continuous on $[a, b]$, is of bounded variation on $[a, b]$.

Proof. Since f is absolutely continuous over $[a, b]$, there is a $\delta > 0$ so that for any finite collection $\{[a_k, b_k]\}_{k=1}^n$ of subintervals of $[a, b]$ so that $\sum_{k=1}^n (b_k - a_k) < \delta$, $\sum_{k=1}^n |f(b_k) - f(a_k)| < 1$.
Let $Q = \{x_0 = a, x_1, ..., x_n = b\}$ be a partition so that $x_k - x_{k-1} < \delta$ for any k.

Let P be any partition of $[a, b]$. Consider $P \cup Q$, and for each k, let P_k be the sub-partition of $P \cup Q$ in $[x_{k-1}, x_k]$. Then we have

$$V(f, P) \leq V(f, P \cup Q) = \sum_{k=1}^{n} V(f, P_k) < \sum_{k=1}^{n} 1 = n.$$

This n is independent of the choice of partition of $[a, b]$. Therefore $TV(f) < n$. Hence f is of bounded variation. □

Problem RA4: Suppose $E \subset R$ is measurable, and there is $f \in L^1(E)$ so that $f > 0$ almost everywhere, and $\int_E f = 0$. Show $m(E) = 0$.

Proof. Refer to my June 2019 prelim solutions. □