Problem AC 1: Let $L \in \mathbb{R}$. A sequence $\{a_n\}$ is said to have Cesaro sum L if the partial sum sequence $\{s_N\}$ where $s_N = \sum_{n=1}^{N} a_n$ satisfies

$$\lim_{N\to\infty} \frac{s_1 + s_2 + ... + s_N}{N} = L.$$

Show that if

$$L = \sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} s_N,$$

then $\{a_n\}$ has Cesaro sum L.

Proof. Let $\varepsilon > 0$. Then there is a $K \in \mathbb{N}$ so that $|s_N - L| \leq \varepsilon$ whenever $N \geq K$. Let $M = \sum_{n=1}^{K-1} |s_n - L|$.
Let $N \geq K$. Consider

$$\left| \frac{\sum_{n=1}^{N} s_n}{N} - L \right| = \left| \frac{\sum_{n=1}^{N} s_n - \sum_{n=1}^{N} L}{N} \right| \leq \frac{\sum_{n=1}^{N} |s_n - L|}{N}$$
$$= \frac{M}{N} + \frac{\sum_{n=K}^{N} |s_n - L|}{N} \leq \frac{M}{N} + \frac{(N-K+1)}{N}\varepsilon \leq \frac{M}{N} + \varepsilon\,.$$

Since $\varepsilon > 0$ arbitrary, we have

$$0 \leq \lim_{N\to\infty} \left| \frac{\sum_{n=1}^{N} s_n}{N} - L \right| \leq \lim_{N\to\infty} \frac{M}{N} = 0.$$

Thus $\lim_{N\to\infty} \frac{\sum_{n=1}^{N} s_n}{N} = L$. □

Problem AC 2: Suppose $\{f_n\}$ is a sequence of continuous functions on $[0,1]$ so that $f_n \to f$ pointwise on $[0,1]$, where f is also continuous on $[0,1]$. Is it true that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx \to \int_0^1 f(x)dx?$$

Soln. Nope!
Consider the sequence $\{f_n\}$ defined via

$$f_n(x) = \begin{cases} 4n^2 x & , 0 \leq x \leq \frac{1}{2n} \\ -4n^2 x + 4n & , \frac{1}{2n} < x < \frac{1}{n} \\ 0 & , else \end{cases}$$

Then each f_n is continuous on $[0,1]$, the sequence converges pointwise to $f \equiv 0$, which is continuous on $[0,1]$. But yet

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \lim_{n\to\infty} 1 = 1 \neq 0 = \int_0^1 f(x)dx.$$

□

Problem AC 3: Let $\{f_n : E \subset \mathbb{R} \to \mathbb{R}\}$ be a sequence of functions. Show that $\{f_n\}$ converges uniformly on E if and only if for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ so that

$$\|f_n - f_m\| = \sup_{x \in E} |f_n(x) - f_m(x)| < \varepsilon$$

whenever $m, n \geq N$.

Proof. Please note that saying $f_n \to f$ uniformly in E is the same as saying f is function on E and $\|f_n - f\| = \sup_{x\in E} |f_n(x) - f(x)| \to 0$ as $n \to \infty$. Also it is not difficult to show $\|\cdot\|$ is a norm on the real-valued functions on E.
($\Rightarrow$) Let $\varepsilon > 0$ and $f : E \to \mathbb{R}$ be the function to which the sequence converges uniformly. Then there is N so that $\|f_n - f\| < \varepsilon/2$ whenever $n \geq N$.
Let $m, n \geq N$. Then

$$\|f_n - f_m\| \leq \|f_n - f\| + \|f - f_m\| < 2 \cdot (\varepsilon/2) = \varepsilon\,.$$

($\Leftarrow$) Let $\varepsilon > 0$ and let $x \in E$. Then there is an N so that whenever $m, n \geq N$,

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\| < \varepsilon\,.$$

Therefore, for any $x \in E$, $\{f_n(x)\}$ is a Cauchy sequence of real numbers. Therefore we may define $f : E \to \mathbb{R}$ via $f(x) = \lim_{n\to\infty} f_n(x)$.
<u>Claim:</u> $f_n \to f$ uniformly in E.
Let $m, n \geq N$. Then for any $x \in E$,

$$|f_n(x)-f(x)| \leq |f_n(x)-f_m(x)|+|f_m(x)-f(x)| \leq \|f_n-f_m\|+|f_m(x)-f(x)| < \varepsilon+|f_m(x)-f(x)|.$$

Now if we let $m \to \infty$, we get

$$|f_n(x) - f(x)| < \varepsilon, \quad \forall x \in E.$$

Therefore by definitition of supremum,

$$\|f_n - f\| < \varepsilon\,.$$

Therefore we have uniform convergence in E. □

Problem AC 4: Let f be a continuous real-valued function on $[a, b]$. Suppose $a < b$. Prove that

$$\lim_{p\to\infty} \left(\int_a^b |f(x)|^p dx \right)^{1/p} = \max_{x\in[a,b]} |f(x)|.$$

Proof. Since f is continuous and $[a, b]$ is compact, there is a $c \in [a, b]$ so that $|f(c)| = \max_{x\in[a,b]} |f(x)|$.
In one direction,

$$\lim_{p\to\infty} \left(\int_a^b |f(x)|^p dx \right)^{1/p} \leq \lim_{p\to\infty} \left(\int_a^b |f(c)|^p dx \right)^{1/p} = |f(c)| \lim_{p\to\infty} (b - a)^{1/p} = |f(c)|.$$

On the other hand, since f is continuous on $[a,b]$, it is, in particular, continuous at $c \in [a,b]$. Let $\varepsilon > 0$. Then there is $\delta > 0$ so that $\Big||f(x)| - |f(c)|\Big| \leq |f(x) - f(c)| \leq \varepsilon$ whenever $|x-c| < \delta$. Therefore

$$\lim_{p\to\infty}\Big(\int_a^b |f(x)|^p dx\Big)^{1/p} \geq \lim_{p\to\infty}\Big(\int_{c-\frac{\delta}{2}}^{c+\frac{\delta}{2}} |f(x)|^p dx\Big)^{1/p} \geq \lim_{p\to\infty}\Big(\int_{c-\frac{\delta}{2}}^{c+\frac{\delta}{2}} (|f(c)|-\varepsilon)^p dx\Big)^{1/p}$$
$$= (|f(c)| - \varepsilon)\lim_{p\to\infty}\delta^{1/p} = |f(c)| - \varepsilon\,.$$

Since ε is arbitrary, we have

$$\lim_{p\to\infty}\Big(\int_a^b |f(x)|^p dx\Big)^{1/p} \geq |f(c)|.$$

□

Transition Problem: Let f be a measurable function on $[0,1]$. Suppose that

$$M = \inf\{\alpha \in \mathbb{R} : m(\{x \in [0,1] : |f(x)| > \alpha\}) = 0\} < \infty.$$

Show that $|f(x)| \leq M$ for a.e. $x \in [0,1]$ and

$$\lim_{p\to\infty}\Big(\int_0^1 |f|^p\Big)^{1/p} = M.$$

Proof. For each $\alpha \in \mathbb{R}$, let $E_\alpha = \{x \in [0,1] : |f(x)| > \alpha\}$. Let $U = \{\alpha \in \mathbb{R} : m(E_\alpha) = 0\}$. Then $M = \inf U$.
These $\alpha \in U$, are called *essential upper bounds* of f on $[0,1]$, and M is called the *essential supremum* of f on $[0,1]$. This is analogous to the max of the absolute value of a continuous function f on $[0,1]$, like in **Problem AC 4** . Also note that since f is measurable E_α is measurable for each $\alpha \in \mathbb{R}$.
First let us prove that $|f(x)| \leq M$ for a.e. $x \in [0,1]$. By definition of infimum, for every $n \in \mathbb{N}$, there is $\alpha_n \in U$ so that $M \leq \alpha_n < M - (1/n)$. Let $A = \{x \in [0,1] : |f(x)| > M\}$. Then for each $x \in A$, there is $\alpha_n \in U$ so that $|f(x)| > \alpha_n$. Then $x \in E_{\alpha_n}$. Therefore $A \subset \bigcup_{n=1}^\infty E_{\alpha_n}$, and so $m(A) \leq \sum_{n=1}^\infty m(E_{\alpha_n}) = 0$. Hence $m(A) = 0$.
With this we see

$$\lim_{p\to\infty}\Big(\int_0^1 |f|^p\Big)^{1/p} \leq \lim_{p\to\infty}\Big(\int_0^1 M^p\Big)^{1/p} = \lim_{p\to\infty}\Big(M^p\Big)^{1/p} = M.$$

Now let's look to the other direction. Let $E = \{x \in [0,1] : |f(x)| \leq M\}$ and $Z = [0,1] \setminus E$. By using the definition of M, we see $M = \sup_{x\in E}|f(x)|$.
Let $\varepsilon > 0$. Then by definition of supremum, there is a subset $E' = \{x \in E : M \geq |f(x)| > M - \varepsilon\}$. Moreover, $m(E') > 0$.

To see this, suppose $m(E') = 0$. Then $E_{M-\varepsilon} = Z \cup E'$ has measure zero and $M - \varepsilon \in U$. But this is impossible since $M - \varepsilon < M$. Therefore $m(E') > 0$.
With this, we have

$$\lim_{p\to\infty}\Big(\int_0^1 |f|^p\Big)^{1/p} \geq \lim_{p\to\infty}\Big(\int_{E'} |f|^p\Big)^{1/p} > \lim_{p\to\infty}\Big(\int_{E'} |M-\varepsilon\,|^p\Big)^{1/p} = |M-\varepsilon\,| \lim_{p\to\infty}\big(m(E')\big)^{1/p}$$
$$= |M-\varepsilon\,|.$$

Since we can take ε arbitrarily small, we may say

$$\lim_{p\to\infty}\Big(\int_0^1 |f|^p\Big)^{1/p} \geq M.$$

Hence

$$\lim_{p\to\infty}\Big(\int_0^1 |f|^p\Big)^{1/p} = M.$$

□

Problem RA 1: Let $A = [0,1]\times[0,1]$ and $f_n : A \to \mathbb{R}$ be a uniformly bounded sequence of measurable functions so that, for a.e. $x \in [0,1]$, $f_n(x,y) \to xy$, pointwise in y. Let μ denote Lebesgue measure on $\mathbb{R}^2$. Show

$$\lim_{n\to\infty}\int_A f_n d\mu$$

exists, and compute the limit.

Proof. Note that we have a uniformly bounded sequence of measurable functions defined on a set of finite measure. This is just screaming bounded convergence theorem. However we are missing something: We are given a pointwise convergence in y for a.e. $x \in [0,1]$, but we require the a.e. pointwise convergence to be in A.
Let $E = \{x \in [0,1] : f_n(x,y) \not\to xy\}$. Then $m(E) = 0$. Then let $\tilde{E} = \{(x,y) \in A : f_n(x,y) \not\to xy\}$. We can see

$$\tilde{E} \subset E \times [0,1].$$

Then by product measure, $\mu(\tilde{E}) \leq m(E)\cdot 1 = 0$. Therefore we have the a.e. convergence in A.
Now it isn't difficult to show that $f(x,y) = xy$ is continuous on A. (Just show $g(x,y) = x$ and $h(x,y) = y$ are continuous on A. Then f is a product of continuous functions.) Since A is compact, f is Lebesgue integrable on A. Therefore we may use Fubini along with BCT to get

$$\lim_{n\to\infty}\int_A f_n d\mu = \int_A xy d\mu = \int_0^1\Big(\int_0^1 xydx\Big)dy = \int_0^1 (\frac{1}{2}y)dy = \frac{1}{4}.$$

□

Problem RA 2: Suppose $A \subset \mathbb{R}$ has positive outer measure. Let $0 < \alpha < 1$. Show that there exists an interval I such that

$$m^*(A \cap I) \geq \alpha m^*(I).$$

Proof. Suppose for contradiction that for every interval I, $m^*(A \cap I) < \alpha m^*(I)$.
Let $\varepsilon > 0$. Then there is a collection $\{I_j\}_{j=1}^{\infty}$ of intervals so that $A \subset \bigcup_{j=1}^{\infty} I_j$ and $\sum_{j=1}^{\infty} m^*(I_j) < m^*(A) + (\varepsilon/\alpha)$.
Consider

$$m^*(A) = m^*\Big(A \cap (\bigcup_{j=1}^{\infty} I_j)\Big) = m^*\Big(\bigcup_{j=1}^{\infty}(A \cap I_j)\Big) \leq \sum_{j=1}^{\infty} m^*(A \cap I_j) < \alpha \sum_{j=1}^{\infty} m^*(I_j) < \alpha m^*(A) + \varepsilon\,.$$

Since we may take $\varepsilon > 0$ arbitraritly small, we have

$$m^*(A) \leq \alpha m^*(A).$$

But this is not possible since $m^*(A) > 0$ and $0 < \alpha < 1$.
Therefore there must exists I so that $m^*(A \cap I) \geq \alpha m^*(I)$. □

Problem RA 3: Suppose $f, g : [0,1] \to \mathbb{R}$ are absolutely continuous functions. Show that their product fg is also absolutely continuous on $[0,1]$.

Proof. Note that f, g are continuous on $[0,1]$ and $[0,1]$ is compact. Therefore there exists $M = max_{x\in[0,1]}|f(x)|$ and $N = max_{x\in[0,1]}|g(x)|$.
Let $\varepsilon > 0$. Since f, g are absolutely continuous, let $\delta_1 > 0$ correspond to f and $(\varepsilon/2N)$, and let $\delta_2 > 0$ correspond to g and $(\varepsilon/2M)$. Let $\delta = min\{\delta_1, \delta_2\}$.
Let $\{[a_k, b_k]\}_{k=1}^{n}$ be a finite collection of disjoint sub-intervals of $[0,1]$ and suppose $\sum_{k=1}^{n}(b_k - a_k) < \delta$.
Then

$$\begin{aligned}
\sum_{k=1}^{n} |f(a_k)g(a_k) - f(b_k)g(b_k)| &= \sum_{k=1}^{n} |f(a_k)g(a_k) - f(b_k)g(a_k) + f(b_k)g(a_k) - f(b_k)g(b_k)| \\
&\leq \sum_{k=1}^{n} |g(a_k)||f(a_k) - f(b_k)| + \sum_{k=1}^{n} |f(b_k)||g(a_k) - g(b_k)| \\
&\leq N\sum_{k=1}^{n} |f(a_k) - f(b_k)| + M\sum_{k=1}^{n} |g(a_k) - g(b_k)| \\
&< N(\varepsilon/2N) + M(\varepsilon/2M) = \varepsilon\,.
\end{aligned}$$

□