**Problem AC 1:** Let $L \in \mathbb{R}$. A sequence \( \{a_n\} \) is said to have Cesaro sum $L$ if the partial sum sequence \( \{s_N\} \) where $s_N = \sum_{n=1}^{N} a_n$ satisfies
\[
\lim_{N \to \infty} \frac{s_1 + s_2 + \ldots + s_N}{N} = L.
\]
Show that if $L = \sum_{n=1}^{\infty} a_n = \lim_{N \to \infty} s_N$, then \( \{a_n\} \) has Cesaro sum $L$.

**Proof.** Let $\varepsilon > 0$. Then there is a $K \in \mathbb{N}$ so that $|s_N - L| \leq \varepsilon$ whenever $N \geq K$. Let $M = \sum_{n=1}^{K-1} |s_n - L|$. Let $N \geq K$. Consider
\[
\left| \frac{\sum_{n=1}^{N} s_n}{N} - L \right| = \left| \frac{\sum_{n=1}^{N} s_n - \sum_{n=1}^{N} L}{N} \right| \leq \sum_{n=1}^{N} \frac{|s_n - L|}{N} \leq \frac{M}{N} + \left( \frac{N - K + 1}{N} \right) \varepsilon \leq \frac{M}{N} + \varepsilon.
\]
Since $\varepsilon > 0$ arbitrary, we have
\[
0 \leq \lim_{N \to \infty} \left| \frac{\sum_{n=1}^{N} s_n}{N} - L \right| \leq \lim_{N \to \infty} \frac{M}{N} = 0.
\]
Thus $\lim_{N \to \infty} \frac{\sum_{n=1}^{N} s_n}{N} = L$. \( \square \)

**Problem AC 2:** Suppose \( \{f_n\} \) is a sequence of continuous functions on $[0, 1]$ so that $f_n \to f$ pointwise on $[0, 1]$, where $f$ is also continuous on $[0, 1]$. Is it true that
\[
\lim_{n \to \infty} \int_{0}^{1} f_n(x)dx \to \int_{0}^{1} f(x)dx?
\]

**Soln.** Nope!

Consider the sequence \( \{f_n\} \) defined via
\[
f_n(x) = \begin{cases} 
4n^2x & , 0 \leq x \leq \frac{1}{2n} \\
-4n^2x + 4n & , \frac{1}{2n} < x < \frac{1}{n} \\
0 & , else
\end{cases}
\]

Then each $f_n$ is continuous on $[0, 1]$, the sequence converges pointwise to $f \equiv 0$, which is continuous on $[0, 1]$. But yet
\[
\lim_{n \to \infty} \int_{0}^{1} f_n(x)dx = \lim_{n \to \infty} 1 = 1 \neq 0 = \int_{0}^{1} f(x)dx.
\]
\( \square \)
Problem AC 3: Let \( \{f_n : E \subseteq \mathbb{R} \to \mathbb{R}\} \) be a sequence of functions. Show that \( \{f_n\} \) converges uniformly on \( E \) if and only if for every \( \varepsilon > 0 \) there is an \( N \in \mathbb{N} \) so that
\[
\|f_n - f_m\| = \sup_{x \in E} |f_n(x) - f_m(x)| < \varepsilon
\]
whenever \( m, n \geq N \).

Proof. Please note that saying \( f_n \to f \) uniformly in \( E \) is the same as saying \( f \) is function on \( E \) and \( \|f_n - f\| = \sup_{x \in E} |f_n(x) - f(x)| \to 0 \) as \( n \to \infty \). Also it is not difficult to show \( \|\cdot\| \) is a norm on the real-valued functions on \( E \).

(\(\Rightarrow\)) Let \( \varepsilon > 0 \) and \( f : E \to \mathbb{R} \) be the function to which the sequence converges uniformly. Then there is a \( N \) so that \( \|f_n - f\| < \varepsilon/2 \) whenever \( n \geq N \).

Let \( m, n \geq N \). Then
\[
\|f_n - f_m\| \leq \|f_n - f\| + \|f - f_m\| < 2 \cdot (\varepsilon/2) = \varepsilon.
\]

(\(\Leftarrow\)) Let \( \varepsilon > 0 \) and let \( x \in E \). Then there is an \( N \) so that whenever \( m, n \geq N \),
\[
|f_n(x) - f_m(x)| \leq \|f_n - f_m\| < \varepsilon.
\]
Therefore, for any \( x \in E \), \( \{f_n(x)\} \) is a Cauchy sequence of real numbers. Therefore we may define \( f : E \to \mathbb{R} \) via \( f(x) = \lim_{n \to \infty} f_n(x) \).

Claim: \( f_n \to f \) uniformly in \( E \).

Let \( m, n \geq N \). Then for any \( x \in E \),
\[
|f_n(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| \leq \|f_n - f_m\| + |f_m(x) - f(x)| < \varepsilon + |f_m(x) - f(x)|.
\]

Now if we let \( m \to \infty \), we get
\[
|f_n(x) - f(x)| < \varepsilon, \quad \forall x \in E.
\]

Therefore by definitition of supremum,
\[
\|f_n - f\| < \varepsilon.
\]
Therefore we have uniform convergence in \( E \).

\[\square\]

Problem AC 4: Let \( f \) be a continuous real-valued function on \([a, b]\). Suppose \( a < b \). Prove that
\[
\lim_{p \to \infty} \left( \int_a^b |f(x)|^p \, dx \right)^{1/p} = \max_{x \in [a, b]} |f(x)|.
\]

Proof. Since \( f \) is continuous and \([a, b]\) is compact, there is a \( c \in [a, b] \) so that \( |f(c)| = \max_{x \in [a, b]} |f(x)| \).

In one direction,
\[
\lim_{p \to \infty} \left( \int_a^b |f(x)|^p \, dx \right)^{1/p} \leq \lim_{p \to \infty} \left( \int_a^b |f(c)|^p \, dx \right)^{1/p} = |f(c)| \lim_{p \to \infty} (b - a)^{1/p} = |f(c)|.
\]
On the other hand, since \( f \) is continuous on \([a, b]\), it is, in particular, continuous at \( c \in [a, b] \). Let \( \varepsilon > 0 \). Then there is \( \delta > 0 \) so that \(|f(x) - f(c)| \leq \varepsilon \) whenever \(|x - c| < \delta \). Therefore

\[
\lim_{p \to \infty} \left( \int_a^b |f(x)|^p \, dx \right)^{1/p} \geq \lim_{p \to \infty} \left( \int_{c-\frac{\delta}{2}}^{c+\frac{\delta}{2}} |f(x)|^p \, dx \right)^{1/p} \geq \lim_{p \to \infty} \left( \int_{c-\frac{\delta}{2}}^{c+\frac{\delta}{2}} (|f(c)| - \varepsilon)^p \, dx \right)^{1/p} = (|f(c)| - \varepsilon) \lim_{p \to \infty} \delta^{1/p} = |f(c)| - \varepsilon.
\]

Since \( \varepsilon \) is arbitrary, we have

\[
\lim_{p \to \infty} \left( \int_a^b |f(x)|^p \, dx \right)^{1/p} \geq |f(c)|.
\]

\[\square\]

**Transition Problem:** Let \( f \) be a measurable function on \([0, 1]\). Suppose that

\[
M = \inf \{ \alpha \in \mathbb{R} : m(\{x \in [0, 1] : |f(x)| > \alpha \}) = 0 \} < \infty.
\]

Show that \(|f(x)| \leq M\) for a.e. \( x \in [0, 1] \) and

\[
\lim_{p \to \infty} \left( \int_0^1 |f|^p \, dx \right)^{1/p} = M.
\]

**Proof.** For each \( \alpha \in \mathbb{R} \), let \( E_\alpha = \{ x \in [0, 1] : |f(x)| > \alpha \} \). Let \( U = \{ \alpha \in \mathbb{R} : m(E_\alpha) = 0 \} \).

Then \( M = \inf U \).

These \( \alpha \in U \), are called *essential upper bounds* of \( f \) on \([0, 1]\), and \( M \) is called the *essential supremum* of \( f \) on \([0, 1]\). This is analogous to the max of the absolute value of a continuous function \( f \) on \([0, 1]\), like in **Problem AC 4**. Also note that since \( f \) is measurable \( E_\alpha \) is measurable for each \( \alpha \in \mathbb{R} \).

First let us prove that \(|f(x)| \leq M\) for a.e. \( x \in [0, 1] \). By definition of infimum, for every \( n \in \mathbb{N} \), there is \( \alpha_n \in U \) so that \( M \leq \alpha_n < M - (1/n) \). Let \( A = \{ x \in [0, 1] : |f(x)| > M \} \). Then for each \( x \in A \), there is \( \alpha_n \in U \) so that \(|f(x)| > \alpha_n \). Then \( x \in E_{\alpha_n} \). Therefore \( A \subset \bigcup_{n=1}^\infty E_{\alpha_n} \), and so \( m(A) \leq \sum_{n=1}^\infty m(E_{\alpha_n}) = 0 \). Hence \( m(A) = 0 \).

With this we see

\[
\lim_{p \to \infty} \left( \int_0^1 |f|^p \, dx \right)^{1/p} \leq \lim_{p \to \infty} \left( \int_0^1 M^p \, dx \right)^{1/p} = \lim_{p \to \infty} \left( M^p \right)^{1/p} = M.
\]

Now let’s look to the other direction. Let \( E = \{ x \in [0, 1] : |f(x)| \leq M \} \) and \( Z = [0, 1] \setminus E \).

By using the definition of \( M \), we see \( M = \sup_{x \in E} |f(x)| \).

Let \( \varepsilon > 0 \). Then by definition of supremum, there is a subset \( E' = \{ x \in E : M \geq |f(x)| > M - \varepsilon \} \). Moreover, \( m(E') > 0 \).
To see this, suppose \( m(E') = 0 \). Then \( E_{M-\varepsilon} = Z \cup E' \) has measure zero and \( M - \varepsilon \in U \). But this is impossible since \( M - \varepsilon < M \). Therefore \( m(E') > 0 \).

With this, we have

\[
\lim_{p \to \infty} \left( \int_{0}^{1} |f|^p \right)^{1/p} \geq \lim_{p \to \infty} \left( \int_{E'} |f|^p \right)^{1/p} > \lim_{p \to \infty} \left( \int_{E'} |M - \varepsilon|^p \right)^{1/p} = |M - \varepsilon| \lim_{p \to \infty} \left( m(E') \right)^{1/p} = |M - \varepsilon|.
\]

Since we can take \( \varepsilon \) arbitrarily small, we may say

\[
\lim_{p \to \infty} \left( \int_{0}^{1} |f|^p \right)^{1/p} \geq M.
\]

Hence

\[
\lim_{p \to \infty} \left( \int_{0}^{1} |f|^p \right)^{1/p} = M.
\]

\[\square\]

**Problem RA 1:** Let \( A = [0, 1] \times [0, 1] \) and \( f_n : A \to \mathbb{R} \) be a uniformly bounded sequence of measurable functions so that, for a.e. \( x \in [0, 1] \), \( f_n(x, y) \to xy \), pointwise in \( y \). Let \( \mu \) denote Lebesgue measure on \( \mathbb{R}^2 \). Show

\[
\lim_{n \to \infty} \int_{A} f_n d\mu
\]

exists, and compute the limit.

**Proof.** Note that we have a uniformly bounded sequence of measurable functions defined on a set of finite measure. This is just screaming bounded convergence theorem. However we are missing something: We are given a pointwise convergence in \( y \) for a.e. \( x \in [0, 1] \), but we require the a.e. pointwise convergence to be in \( A \).

Let \( E = \{ x \in [0, 1] : f_n(x, y) \not\to xy \} \). Then \( m(E) = 0 \). Then let \( \tilde{E} = \{ (x, y) \in A : f_n(x, y) \not\to xy \} \). We can see

\[
\tilde{E} \subset E \times [0, 1].
\]

Then by product measure, \( \mu(\tilde{E}) \leq m(E) \cdot 1 = 0 \). Therefore we have the a.e. convergence in \( A \).

Now it isn’t difficult to show that \( f(x, y) = xy \) is continuous on \( A \). (Just show \( g(x, y) = x \) and \( h(x, y) = y \) are continuous on \( A \). Then \( f \) is a product of continuous functions.) Since \( A \) is compact, \( f \) is Lebesgue integrable on \( A \). Therefore we may use Fubini along with BCT to get

\[
\lim_{n \to \infty} \int_{A} f_n d\mu = \int_{A} xy d\mu = \int_{0}^{1} \left( \int_{0}^{1} xy dx \right) dy = \int_{0}^{1} \left( \frac{1}{2} y \right) dy = \frac{1}{4}.
\]

\[\square\]
**Problem RA 2:** Suppose $A \subset \mathbb{R}$ has positive outer measure. Let $0 < \alpha < 1$. Show that there exists an interval $I$ such that

$$m^*(A \cap I) \geq \alpha m^*(I).$$

**Proof.** Suppose for contradiction that for every interval $I$, $m^*(A \cap I) < \alpha m^*(I)$.

Let $\varepsilon > 0$. Then there is a collection $\{I_j\}_{j=1}^\infty$ of intervals so that $A \subset \bigcup_{j=1}^\infty I_j$ and $\sum_{j=1}^\infty m^*(I_j) < m^*(A) + (\varepsilon / \alpha)$.

Consider

$$m^*(A) = m^*(A \cap \bigcup_{j=1}^\infty I_j) = m^*(\bigcup_{j=1}^\infty (A \cap I_j)) \leq \sum_{j=1}^\infty m^*(A \cap I_j) < \alpha \sum_{j=1}^\infty m^*(I_j) < \alpha m^*(A) + \varepsilon.$$

Since we may take $\varepsilon > 0$ arbitraritly small, we have

$$m^*(A) \leq \alpha m^*(A).$$

But this is not possible since $m^*(A) > 0$ and $0 < \alpha < 1$.

Therefore there must exists $I$ so that $m^*(A \cap I) \geq \alpha m^*(I)$. \hfill \Box

**Problem RA 3:** Suppose $f, g : [0, 1] \to \mathbb{R}$ are absolutely continuous functions. Show that their product $fg$ is also absolutely continuous on $[0, 1]$.

**Proof.** Note that $f, g$ are continuous on $[0, 1]$ and $[0, 1]$ is compact. Therefore there exists $M = \max_{x \in [0, 1]} |f(x)|$ and $N = \max_{x \in [0, 1]} |g(x)|$.

Let $\varepsilon > 0$. Since $f, g$ are absolutely continuous, let $\delta_1 > 0$ correspond to $f$ and $(\varepsilon / 2N)$, and let $\delta_2 > 0$ correspond to $g$ and $(\varepsilon / 2M)$. Let $\delta = \min\{\delta_1, \delta_2\}$.

Let $\{[a_k, b_k]\}_{k=1}^n$ be a finite collection of disjoint sub-intervals of $[0, 1]$ and suppose $\sum_{k=1}^n (b_k - a_k) < \delta$.

Then

$$\sum_{k=1}^n |f(a_k)g(a_k) - f(b_k)g(b_k)| = \sum_{k=1}^n |f(a_k)g(a_k) - f(b_k)g(a_k) + f(b_k)g(a_k) - f(b_k)g(b_k)|$$

$$\leq \sum_{k=1}^n |g(a_k)||f(a_k) - f(b_k)| + \sum_{k=1}^n |f(b_k)||g(a_k) - g(b_k)|$$

$$\leq N \sum_{k=1}^n |f(a_k) - f(b_k)| + M \sum_{k=1}^n |g(a_k) - g(b_k)|$$

$$< N(\varepsilon / 2N) + M(\varepsilon / 2M) = \varepsilon.$$ \hfill \Box

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