The Davey-Stewartson II Equation: Dispersive Asymptotics, Soliton Solutions

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Bill of Fare

Notation (1 of 3): Spaces

$C^0(\mathbb{C})$ The Banach space of continuous functions f on $\mathbb{C}$ with $\|f\|_{C^0(\mathbb{C}} = \sup_{z\in\mathbb{C}} |f(z)|$ finite

$H^{1,1}(\mathbb{C})$ The Hilbert space of $L^2(\mathbb{C})$ functions u with $(1+|\cdot|)u(\cdot)$ and ∇u in $L^2(\mathbb{C})$

$L^p(L^q)$ The space of functions $a(z,\zeta)$ with

$$\left(\int \left| \int |a(z,\zeta)|^q \, d\zeta \right|^{1/q} dA(z) \right)^{1/p}$$

Notation (2 of 3): Transforms

$e_k(z)$ The function $\exp i\left(kz + \overline{k}\overline{z}\right)$

$\mathcal{F}$ The Fourier transform

$$(\mathcal{F}\psi)(k) = \frac{1}{\pi}\int e_k(z)\psi(z)\,dA(z)$$

$\mathcal{C}$ The solid Cauchy transform

$$\mathcal{C}\left[f\right](z) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{1}{z-\zeta} f(\zeta)\,dA(\zeta)$$

$\mathcal{C}_k$ The solid Cauchy transform in the k variable

$$\mathcal{C}_k\left[f\right](k) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{1}{k-\zeta} f(\zeta)\,dA(\zeta)$$

Notation (3 of 3): Operators

If X is a Banach space, $\mathcal{B}(X)$ denotes the bounded linear operators from X to itself.

$\mathcal{B}(X)$ is itself a Banach space with norm

$$\|T\|_{\mathcal{B}(X)} = \sup\left\{\|Tx\|_X : \|x\|_X = 1\right\}$$

The *compact operators* are the norm-closure of finite-rank operators in $\mathcal{B}(X)$. They are important because $(I+T)$ is Fredholm

For certain subalgebras $\mathcal{A}$ of the compact operators, one can define a determinant $\det(I+T)$ for $T \in \mathcal{A}$ with the property that $(I+T)^{-1}$ exists iff $\det(I+T) \neq 0$

The Mikhlin-Itskovich Algebra (1 of 2)

The Mikhlin-Itskovich algebra $\mathcal{A}_p$ is an algebra of compact integral operators on $L^p(\mathbb{C})$.

Fix $p \in (1, \infty)$. The operator

$$(Af)(z) = \int a(z, \zeta) f(\zeta)\, dA(\zeta)$$

belongs to $\mathcal{A}_p$ if

- The integral kernel $a(z, \zeta)$ belongs to $L^p(L^q)$ where $p^{-1} + q^{-1} = 1$
- The integral kernel $a^*(z, \zeta) = \overline{a(\zeta, z)}$ belongs to $L^q(L^p)$ for the same (p, q)

The Mikhlin-Itskovich Algebra (2 of 2)

The norm on $\mathcal{A}_p$ is

$$\|A\|_{\mathcal{A}_p} = \max\left(\|a\|_{L^p(L^q)}, \|a^*\|_{L^p(L^q)}\right)$$

For any $A \in \mathcal{A}_p$,

$$\|A\|_{\mathcal{B}(L^p)} \leq \|A\|_{\mathcal{A}_p}$$

The finite-rank operators are dense in $\mathcal{A}_p$ and the norm satisfies

$$\|AB\|_{\mathcal{A}_p} \leq \|A\|_{\mathcal{A}_p} \|B\|_{\mathcal{A}_p}.$$

Renormalized Determinants on $\mathcal{A}_p$

Gohberg, Goldberg, and Krein (1997) defined a renormalized determinant $\mathrm{Det}(I+A)$ for $A \in \mathcal{A}_p$ with the following properties:

- If F is a finite-rank operator,

$$\mathrm{Det}(I+F) = \det\left((I+F)e^{-F}\right)$$

- $(I+A)^{-1}$ exists on $\mathcal{B}(L^p)$ if and only if $\mathrm{Det}(I+A) \neq 0$

One should understand the "renormalization" via the identity

$$\begin{aligned} \log\det(I+B) &= \operatorname{tr}\log(I+B) \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \operatorname{tr}\left(B^n\right) \end{aligned}$$

For $A \in \mathcal{A}_p$, $\operatorname{tr}(A^n)$ makes sense for $n \geq 2$.

The Davey-Stewartson II Equation

The DS II equation describes the amplitude envelope $u = u(x, y, t)$ of a monochromatic, weakly nonlinear surface wave:

$$\begin{aligned} iu_t + 2\left(\partial^2 + \overline{\partial}^2\right) u + (g + \overline{g})u &= 0 \\ \overline{\partial} g + \sigma \partial \left(|u|^2\right) &= 0 \end{aligned}$$

where $\sigma = +1$ (defocussing) or -1 (focussing). The second equation determines g in terms of u at each time t. Another way of writing this equation is

$$iu_t + 2\left(\partial^2 + \overline{\partial}^2\right) u + 2\sigma \Re\left(\overline{\partial}^{-1} \partial |u|^2\right) u = 0$$

The DS II equation is a two-dimensional analogue of the cubic NLS in one dimension and is completely integrable.

We will study dispersive behavior of the defocussing DS II equation ($\sigma = +1$) and soliton solutions of the focussing DS II equation ($\sigma = -1$)

Key ideas:

- Use "stationary phase" methods and IST to obtain large-time asymptotics of defocussing DS II
- Use renormalized determinants to study soliton solutions of focussing DS II

Lax Representation (Review)

The defocussing DS II admits a Lax representation $\dot{L} = [A, L]$ where

$$L = -\partial_x - iJ\partial_y + Q$$
$$A = B - Q\partial_y + iJ\partial_y^2$$

where

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & u \\ \overline{u} & 0 \end{pmatrix}$$

The equation $L\psi = 0$ defines a spectral problem, while the equation $\psi_t = A\psi$ implies a law of evolution for scattering data.

Defocussing Case: Dispersion

In this section we'll use the solution formula

$$u(z,t) = \mathcal{I}\left[\exp\left(it\left(\diamond^2 + (\overline{\diamond})^2\right)\right)\mathcal{R}(u_0)(\diamond)\right](z)$$

to prove that the defocussing DS II equation exhibits dispersive behavior by computing L^∞ asymptotics to leading order. We make a slighty stronger assumption that

$$\mathcal{R}u_0 \in H^{1,1}(\mathbb{C}) \cap C^0(\mathbb{C})$$

It is sufficient that

$$u_0 \in H^{1,1}(\mathbb{C}) \cap L^1(\mathbb{C})$$

References:

1. L.-Y. Sung, *J. Nonlinear Sci.* **5** (1995), no. 5, 433–452.
2. O. M. Kiselev, *Differential Equations* **33** (1997), no. 6, 815–823.
3. P. Perry, `arXiV:1110.5589` and *J. Spectral Theory*, to appear.

$\overline{\partial}$ Problem with Parameters (1 of 2)

Recall that if u_0 is the Cauchy data $r = \mathcal{R}u_0$, we compute the solution via the formula

$$u(z,t) = \mathcal{I}\left[\exp\left(it\left(\diamond^2 + (\overline{\diamond})^2\right)\right)\mathcal{R}u_0\right](z)$$

Recall that $\mathcal{I} : r \to u$ is defined by the $\overline{\partial}$-problem

$$\left(\overline{\partial}_k \nu_1\right)(z,k) = -\frac{i}{2}e_k(z)\overline{r(k)}\,\overline{\nu_2(z,k)}$$
$$\left(\overline{\partial}_k \nu_2\right)(z,k) = -\frac{i}{2}e_k(z)\overline{r(k)}\,\overline{\nu_1(z,k)}$$

and the reconstruction formula

$$u(z) = \frac{1}{\pi}\int e_{-k}(z)r(k)\nu_1(z,k)\,dA(z)$$

Now replace $r(k)$ by $r(k)\exp(it(k^2 + \overline{k}^2))$.

$\overline{\partial}$ Problem with Parameters (2 of 2)

Hence, to compute

$$u(z,t) = \mathcal{I}\left[\exp\left(it\left(\diamond^2 + (\overline{\diamond})^2\right)\right)\mathcal{R}u_0\right](z)$$

we solve the $\overline{\partial}$-problem with parameters:

$$\left(\overline{\partial}_k \nu_1\right)(z,k) = -\frac{i}{2} e^{-itS}\overline{r(k)}\,\overline{\nu_2(z,k)}$$
$$\left(\overline{\partial}_k \nu_2\right)(z,k) = -\frac{i}{2} e^{-itS}\overline{r(k)}\,\overline{\nu_1(z,k))}$$

$$\lim_{|k|\to\infty}(\nu_1(z,k),\nu_2(z,k)) = (1,0)$$

where

$$S(z,k,t) = \frac{kz+\overline{k}\overline{z}}{t} + \left(k^2 + \overline{k}^2\right)$$

and recover u from

$$u(z,t) = \frac{1}{\pi}\int e^{itS} r(k)\nu_1(z,k)\,dA(k)$$

Integral Equation

Define

$$M\psi = \frac{1}{2}\mathcal{C}_k\left[e^{-itS}\overline{r}\overline{\psi}\right]$$

Then

$$\begin{aligned} \nu_1 &= 1 + M\nu_2 \\ \nu_2 &= M\nu_1 \end{aligned}$$

and

$$u = \frac{1}{\pi}\int e^{itS(z,k,t)} r(k)\nu_1(z,k)\, dA(k)$$

To obtain large-t asymptotics of u in L^∞, we must obtain large-t asymptotics of ν_1 in L^p

Stationary Phase

S has a single, nondegenerate stationary phase point:

$$S(z,k,t) = \frac{kz + \overline{k}\overline{z}}{t} + \left(k^2 + \overline{k}^2\right),$$

$$S_{\overline{k}}(z,k,t) = 2(k - k_c), \qquad k_c = -\frac{z}{2t},$$

By Bukhgeim's formula, if ψ has support away from k_c,

$$(M\psi)(z,k,t) = \frac{e^{-itS(z,k,t)}}{2itS_{\overline{k}}(k)} \overline{r(k)} \psi(k)$$

$$- \frac{1}{2\pi it} \int \frac{e^{-itS(\zeta)}}{k-\zeta} \overline{\partial}_\zeta \left(S_{\overline{k}}^{-1}(\zeta) \overline{r(k)} f(\zeta) \right) dA(\zeta)$$

so $(M\psi)$ decays in t for ψ localized away from k_c.

An "L^∞ Scattering" Result

We'll show that, up to $o\left(t^{-1}\right)$ remainders in L^∞ norm, $u(z,t)$ equals $v(z,t) = \int e^{itS(z,k,t)} r(k)\, dA(k)$ which solves the linear equation

$$iv_t + \left(\partial^2 + \overline{\partial}^2\right) v = 0$$
$$v(z,0) = \left(\mathcal{F}^{-1} r\right)(z)$$

Localization

Fix z, t and hence $k_c = -z/2t$. Let η be a $C^\infty(\mathbb{C})$ function with $\eta(\zeta) = 1$ for $|\zeta| \leq 1$ and $\eta(\zeta) = 0$ for $|\zeta| \geq 2$. Define

$$\chi(k) = \eta\left(t^{1/4}(k - k_c)\right).$$

Thus:

- χ localizes near the critical point, and χr has shrinking support as $t \to \infty$
- $(1 - \chi)$ localizes away from the critical point where stationary phase methods may be used

Operator Estimates

Recall

$$M\psi = \frac{1}{2}\mathcal{C}_k\left[e^{-itS}\overline{r}\overline{\psi}\right]$$

Writing

$$M\psi = \underbrace{M(\chi\psi)}_{\text{small support}} + \underbrace{M((1-\chi)\psi)}_{\text{stationary phase}}$$

we can prove:

Lemma Given any $\varepsilon > 0$, there is a $\delta > 0$ so that

$$\left\|M^2\psi\right\|_{L^{2+2\delta}} \leq Ct^{\varepsilon-1/2}\left\|r\right\|^2_{H^{1,1}}\left\|\psi\right\|_{L^{2+2\delta}}.$$

Resolvent Estimates, Large-t Expansions (1 of 2)

Recall Given any $\varepsilon > 0$, there is a $\delta > 0$ so that

$$\left\| M^2 \psi \right\|_{L^{2+2\delta}} \leq C t^{\varepsilon - 1/2} \left\| r \right\|_{H^{1,1}}^2 \left\| \psi \right\|_{L^{2+2\delta}}.$$

This means that $(I - M^2)$ is invertible for t large, and that one can obtain large-t expansions with controlled remainder by writing

$$(I - M^2)^{-1} = I + \sum_{j=1}^{N} M^{2j} + M^{2N}(I - M^2)^{-1} M^2.$$

To compute $\nu_1 = (I - M^2)^{-1} 1$ to order t^{-1}, one need use only finitely many terms in the Neumann expansion.

Large-t Asymptotics of $u(z,t)$

Recall that $u(z,t)$ solves DS II with initial data u_0, $v(z,t)$ solves the linear equation with initial data $\mathcal{F}^{-1}\mathcal{R}u_0$. Now use the expansion

$$\nu_1 = 1 + \sum_{j=1}^{N} \left(M^{2j}1\right) + M^{2N}(I - M^2)^{-1}\left(M^2 1\right)$$

in the reconstruction formula

$$u(z,t) = \frac{1}{\pi}\int e^{itS(z,k,t)} r(k)\nu_1(z,k,t)\, dA(k)$$

to obtain an expansion for $u(z,t)$ whose lowest-order term is $v(z,t)$

Resolvent Estimates, Large-t Expansions (2 of 2)

$$
\begin{aligned}
u(z,t) - v(z,t) &= \frac{1}{\pi} \int e^{itS(z,k,t)} r(k) \left(\nu_1(z,k,t) - 1\right) \, dA(k) \\
&= \sum_{j=1}^{N} \frac{1}{\pi} \int e^{itS(z,k,t)} r(k) (M^{2j} 1)(z,k,t) \, dA(k) \\
&+ \int e^{itS(z,k,t)} r(k) \nu_{1,N}(z,k,t) \, dA(k)
\end{aligned}
$$

where

$$
\nu_{1,N}(z,k,t) = \left[(M^{2N} (I - M^2)^{-1} (M^2 1) \right] (z,k,t)
$$

The remainders are estimated using stationary phase and cutoff functions.

Large-Time Asymptotics: The Bigger Picture

- The example of defocussing DS II shows what a "nonlinear stationary phase" method may look like in the $\overline{\partial}$-context
- The method (i.e., strong estimates on fundamental integral operators via stationary phase plus resolvent expansions) should apply to other problems with isolated critical points

Focussing Case: Soliton Solutions

In this section we'll use Fredholm determinants to analyze the one-soliton solution to focussing DS II given by Arkadiev, Progrebkov, and Polivanov.

References:

- V. A. Arkadiev, A. K. Progrebkov, M. C. Polivanov. *Physica D.* **36** (1989), 189-197.
- R. Brown, M. Music, and P. Perry, In preparation.
- E. V. Doktorov, S. B. Leble. A Dressing Method in Mathematical Physics. *Mathematical Physics Studies*, **28**, Springer, Dordrecht, 2007
- J. Villarroel, M. Ablowitz. *SIAM J. Math. Anal.* **34** (2003), no. 6, 1253-1278.

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DS II Spectral Problem Revisited (1 of 3)

We'll formulate the spectral problem and scattering data in a different way:

$$D\psi = \frac{1}{2} Q\psi$$

for a 2×2 matrix-valued function ψ.

$$D = \begin{pmatrix} \overline{\partial} & 0 \\ 0 & \partial \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & u \\ -\overline{u} & 0 \end{pmatrix}$$

and factor

$$\psi = ME, \qquad \psi \sim E \text{ as } |z| \to \infty$$

where

$$E = \begin{pmatrix} e^{ikz} & 0 \\ 0 & e^{-i\overline{k}\overline{z}} \end{pmatrix}$$

We reformulate the spectral problem in terms of M

DS II Spectral Problem Revisited (2 of 3)

$$D = \begin{pmatrix} \overline{\partial} & 0 \\ 0 & \partial \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & u \\ -\overline{u} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We find

$$DM - \frac{i}{2}k\left[\sigma_3, M\right] = \frac{1}{2}QM$$

$$\lim_{|z|\to\infty} M(z,k) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

As usual, it suffices by symmetry to study the first column $(M_{11}, M_{21}) =: (M_1, M_2)$.

DS II Spectral Problem Revisited (3 of 3)

Write

$$\mathbf{M}(z) = (M_1(z), M_2(z))^T .$$

We obtain the mixed ∂-$\overline{\partial}$ system

$$\begin{aligned} \overline{\partial}_z M_1 &= \frac{1}{2} u M_2 \\ (\partial_z + ik) M_2 &= -\frac{1}{2} \overline{u} M_1 \\ \lim_{|z| \to \infty} \mathbf{M}(z) &= (1, 0)^T \end{aligned}$$

A point $k \in \mathbb{C}$ is called a *regular point* if this problem has a unique solution, and a *singular point* otherwise.

Integral Equation (1 of 2)

Let

$$G(k)\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \mathcal{C}\left[uw_2\right] \\ -e_{-k}\overline{\mathcal{C}}\left[e_k\overline{u}w_1\right] \end{pmatrix}$$

and

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The spectral problem is

$$\mathbf{M} = \mathbf{e}_1 + G(k)\mathbf{M}$$

If $u \in L^{2-\varepsilon}(\mathbb{C}) \cap L^{2+\varepsilon}(\mathbb{C})$, the operator $G(k)$ is a compact operator from $C^0(\mathbb{C}) \otimes \mathbb{C}^2$ to itself with $\|G(k)\|_{\mathcal{B}(C^0)} \rightarrow 0$ as $|k| \rightarrow \infty$

Integral Equation (2 of 2)

The equation

$$\mathbf{M} = \mathbf{e}_1 + G(k)\mathbf{M}$$

has a unique solution $\mathbf{M}(z,k)$ for sufficiently large k with

$$\lim_{|z|\to\infty} z\left(\mathbf{M}(z,k) - \mathbf{e}_1 - \frac{1}{z}\begin{pmatrix} s(k) \\ r(k) \end{pmatrix}\right) = \mathbf{0}$$

where

$$s(k) = \frac{i}{2\pi}\int u(z) M_2(z,k)\, dA(z)$$
$$r(k) = \frac{i}{2\pi}\int e_k(z)\overline{u(z)} M_1(z,k)\, dA(z)$$

The first is flow-invariant; the second obeys a linear evolution equation.

Scattering Data

Note that, for $u \in \mathcal{S}(\mathbb{C})$, we have

$$s(k) \sim \sum_{j=0}^{\infty} \frac{s_j}{k^{j+1}}$$

where

$$s_0 = -\frac{i}{2\pi} \int_{\mathbb{C}} |u(z)|^2 \, dA(z)$$

Fredholm Determinant (1 of 4)

Suppose $u \in L^{2-\varepsilon}(\mathbb{C}) \cap L^{2+\varepsilon}(\mathbb{C})$ and iterate the integral equation:

$$\mathbf{M}(z,k) = \mathbf{e}_1 + G(k)\mathbf{e}_1 + G(k)(I - G(k))^{-1}\left(G(k)\mathbf{e}_1\right)$$

- It suffices to study the resolvent $(I - G(k))^{-1}$ as a map from $L^p(\mathbb{C}) \otimes \mathbb{C}^2$ to itself.
- The operator $G(k)$ belongs to the Mikhlin-Itskovich algebra $\mathcal{A}_p$ for any $p > 2$
- $(I - G(k))$ is invertible in $\mathcal{B}(L^p)$ if and only if

  $$D(k) := \mathrm{Det}(I - G(k))$$

  is nonzero (Brown, Music, Perry 2014).

Fredholm Determinant (2 of 4)

Recall $(s(k), r(k))$ are scattering data for $\mathbf{M}(z, k)$, and

$$D(k) := \mathrm{Det}(I - G(k))$$

Then, for $|k|$ sufficiently large,

$$\bar{\partial}_k \log D(k) = \frac{i}{2}\overline{s(k)} - c(k)$$

and

$$\lim_{|k| \to \infty} D(k) = 1.$$

Here

$$c(k) = \frac{i}{4\pi} \int \frac{e_{-k}(z) u(w) e_k(z) \overline{u(z)}}{z - w} \, dA(z, w)$$

Fredholm Determinant (3 of 4)

$$\overline{\partial}_k \log D(k) = \frac{i}{2}\overline{s(k)} - c(k)$$

Where the $\overline{\partial}$ equation comes from:

- Determinant differentiation formula

$$\begin{aligned}\overline{\partial} \log \det \left(I - A(z)\right) &= \overline{\partial}\, \mathrm{Tr} \log \left(I - A(z)\right) \\ &= \mathrm{Tr}\left[\left(I - A(z)\right)^{-1} \overline{\partial} A(z)\right]\end{aligned}$$

- $\overline{\partial}_k G(z)$ is a finite-rank operator

Fredholm Determinant (4 of 4)

$$\overline{\partial} \log D(k) = \frac{i}{2}\overline{s(k)} - c(k)$$

$$c(k) = \frac{i}{4\pi} \int \frac{e_{-k}(z) u(w) e_k(z) \overline{u(z)}}{z - w} \, dA(z, w)$$

$$\frac{i}{2}\overline{s(k)} - c(k) \sim \mathcal{O}\left(|k|^{-2}\right)$$

Hence, in principle, one can “integrate in from infinity” to find the determinant.

One-Soliton Ansatz

Using formal arguments, Arkadiev, Progrebov, and Polivanov showed that if

$$u(z) = \frac{2c_1 e_{k_0}(z)}{|z+z_1|^2 + |c_1|^2}$$

then

$$\mathbf{M}(z,k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{k-k_0} m_{-1}(z)$$

where

$$m_{-1}(z) = \frac{i}{|z+z_1|^2 + |c_1|^2} \begin{pmatrix} \overline{z} + \overline{z_1} \\ \overline{c_1} e_{-k_0}(z) \end{pmatrix}.$$

This solution must be correct for *large* k by uniqueness

One-Soliton Conundrum

Recall

$$\overline{\partial} \log D(k) = \frac{i}{2} \overline{s(k)} - c(k).$$

For large k, one can easily compute from the APP ansatz for u and the APP solution $\mathbf{M}(z, k)$ that

$$s(k) = 2i \frac{1}{k - k_0}, \qquad c(k) = \gamma(k - k_0)$$

where

$$\gamma(\kappa) \sim \frac{1}{\overline{\kappa}} + \mathcal{O}\left(|\kappa|^{-2}\right)$$

and $\gamma(\kappa)$ is regular at $\kappa = 0$. One expects that

$$D(k) \sim \log |k - k_0|^2$$

near $k = k_0$, but is the solution still unique near $k = k_0$?

One-Soliton to One Dimension

Recall

$$u(z) = \frac{2c_1 e_{k_0}(z)}{\left|z + z_1\right|^2 + \left|c_1\right|^2}$$

Without loss we can take $z_1 = 0$ (translate) and $c_1 = 1$ (dilate) so

$$u(z) = \frac{e_{k_0}(z)}{|z|^2 + 1}$$

We will show that $\log D(k) = F(|k - k_0|^2)$ for a function F on $(0, \infty)$

- The $\overline{\partial}$-problem becomes "essentially one dimensional"
- We can literally "integrate in from infinity"

From $\overline{\partial}$-Problem to ODE

Let $F(|k-k_0|^2) = \log D(k)$. Then

$$F'(|k-k_0|^2)(k-k_0) = \frac{1}{\overline{k}-\overline{k_0}} - \gamma(k-k_0)$$

or, setting $t = |k-k_0|^2$,

$$F'(t) = 1/t - g(t)$$

where $g(t) = 1/t + \mathcal{O}\left(t^{-2}\right)$ as $t \to \infty$. Of course, since these calculations are based on the APP solution, they only hold *a priori* for t large.

One-Soliton Bootstrap

We'll show that the APP solution is valid down to $k = k_0$ by showing that $\log D(k)$ is finite for all $k \neq k_0$. Recall $t = |k - k_0|^2$ and $F(|k - k_0|^2) = \log D(k)$.

Let

$$R = \inf\{t \in (0, \infty) : F(t) \text{is finite}\}$$

Suppose that $R \neq 0$. The equation

$$F'(t) = 1/t - g(t)$$

holds on (R, ∞) so

$$\int_{R+\varepsilon}^{\infty} \left(\frac{1}{t} - g(t) \right) \, dt = -F(R + \varepsilon)$$

which shows that F has a finite limit as $t \downarrow R$, a contradiction.

The APP Solution

Thus, we have:

Theorem Let u be the APP solition solution

$$u(z) = \frac{2c_1 e_{k_0}(z)}{|z+z_1|^2 + |c_1|^2}.$$

Then $k = k_0$ is the unique exceptional point and there is a unique solution $\mathbf{M}(z,k)$ given by

$$\mathbf{M}(z,k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{k-k_0} m_{-1}(z)$$

where

$$m_{-1}(z) = \frac{i}{|z+z_1|^2 + |c_1|^2} \begin{pmatrix} \overline{z} + \overline{z_1} \\ \overline{c_1} e_{-k_0}(z) \end{pmatrix}.$$

Determinants: The Larger Picture

Fredholm determinants are not a new idea (see for example Kazeykina's thesis!) but there is much that may be done:

- Brown, Music, and Perry used determinants to show that, for the two-dimensional scattering problem associated to the Novikov-Veselov equation, the exceptional set of a compactly supported potential can consist at most of isolated points and smooth curves with finitely many intersections
- The GGK papers give explicit formulas for $(I - T)^{-1}$ in terms of the renormalized determinant that can be used to study singularities of the scattering solutions and scattering transform
- The determinant depends smoothly on the potential and so may be useful in proving "generic" properties of exceptional sets (e.g. their generic absence!)

Some Open Problems

1. Prove an L^2 scattering result for the defocussing DS II equation
2. Develop "nonlinear stationary phase" for other systems such as the Novikov-Veselov equation and the DS I equation
3. Generalize the one-solition bootstrap to finitely many solitons
4. Identify the Villarroel-Ablowitz index with the multiplicity of zeros of the determinant
5. Analyze (in)stability of soliton solutions using the determinant
6. Prove (or disprove) that the exceptional set of (some class of potentials) consists at most of finitely many points.
7. Prove (or disprove) that the generic $C_0^\infty(\mathbb{C})$ potential u has no exceptional points.