The Defocussing Davey-Stewartson II Equation: Inverse Scattering and Global Existence

Peter A. Perry

University of Kentucky

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Bounded and Compact Linear Operators

If X and Y are Banach spaces, $\mathcal{B}(X, Y)$ denotes the Banach space of linear operators from X to Y. We define

$$\|T\|_{\mathcal{B}(X,Y)} = \sup \{\|Tx\|_Y : \|x\|_X = 1\}$$

We write $\mathcal{B}(X)$ for $\mathcal{B}(X, X)$

An operator $T \in \mathcal{B}(X, Y)$ is *compact* if the image of any bounded subset of X has compact closure in Y

Theorem (BLT) *If $T : \mathcal{D} \to Y$ is bounded on a dense subset $\mathcal{D}$ of X, then T admits a unique extension to a bounded operator from X to Y.*

Finding the Resolvent: Neumann Series (1 of 2)

Let $T \in \mathcal{B}(X)$ and consider the equation

$$(I - T)x = y$$

The formal solution is $x = (I - T)^{-1}y$ where

$$(I - T)^{-1} = \sum_{n=0}^{\infty} T^n$$

This *Neumann series* converges if (root test!)

$$\lim_{n\to\infty} \| T^n \|_{\mathcal{B}(X)}^{1/n} < 1$$

$T \in \mathcal{B}(X)$ is a *Volterra operator* if $\lim_{n\to\infty} \| T^n \|_{\mathcal{B}(X)}^{1/n} = 0$

Finding the Resolvent: Neumann Series (2 of 2)

The canoncial example of a Volterra operator is:

- $(Tf)(x) = \int_0^x f(t)\,dt$ on $C^0[0,1]$. Note that

$$
\begin{aligned}
|(T^n f)(x)| &\leq \int_0^x \int_0^{x_1} \cdots \int_0^{x_{n-1}} |f(x_n)|\,dx_n \cdots dx_1 \\
&\leq \frac{x^n}{n!} \sup_{t\in[0,1]} |f(t)|
\end{aligned}
$$

so that $\|T^n\|_{\mathcal{B}(C^0[0,1])} \leq \dfrac{1}{n!}$ and

$$
\lim_{n\to\infty} \|T^n\|_{\mathcal{B}(C^0[0,1])}^{1/n} = 0
$$

Volterra operators also occur in the direct problem for the cubic NLS in one dimension

Finding the Resolvent: Fredholm Theory

Consider the equation

$$(I - T)x = y$$

where $T \in \mathcal{B}(X, X)$ is compact.

Fredholm Alternative *Either*

(i) *The equation* $(I - T)x = y$ *has a unique solution* x *for every* $y \in X$*, or*

(ii) $\text{Ker}(I - T)$ *is nontrivial.*

In the ISM context, a theorem which asserts that $\text{Ker}(I - T)$ is trivial is called a *vanishing theorem*.

Finding the Resolvent: Continuous Dependence

Second Resolvent Formula If S and T belong to $\mathcal{B}(X)$ and $(I-S)^{-1}$, $(I-T)^{-1}$ exist, then

$$(I-S)^{-1}-(I-T)^{-1}=(I-S)^{-1}(S-T)(I-T)^{-1}$$

This can be used to prove:

Theorem Suppose $t \mapsto F(t)$ is a continuous map from $\mathbb{R}$ into $\mathcal{B}(X)$, and that $(I-F(0))^{-1}$ exists. Then, for some interval I containing 0, $(I-F(t))^{-1}$ exists for $t \in I$ and the map

$$t \mapsto (I-F(t))^{-1}$$

is continuous from I into $\mathcal{B}(X)$.

Continuous and Compact Nonlinear Maps

A nonlinear mapping φ from a Banach space X to a Banach space Y is *locally Lipschitz continuous* if for every bounded subset B of X there is a constant $C = C(B)$ so that

$$\|\varphi(x) - \varphi(y)\|_Y \leq C \|x - y\|_X$$

A nonlinear mapping φ from X to Y is *compact* if the image of any bounded subset of X has compact closure in Y

Theorem (BNLT) *Suppose $\mathcal{D}$ is a dense subset of X and that $\varphi : \mathcal{D} \to Y$ is locally Lipschitz continuous. Then φ admits a unique extension to a locally Lipschitz continuous map from X to Y.*

Some Basic Notation

$H^{1,1}(\mathbb{C})$ The Hilbert space of $L^2(\mathbb{C})$ functions u with $(1+|\cdot|)u(\cdot)$ and ∇u in $L^2(\mathbb{C})$

∂ The operator $\frac{1}{2}(\partial_x - i\partial_y)$

$\overline{\partial}$ The operator $\frac{1}{2}(\partial_x + i\partial_y)$

$\mathcal{C}[f]$ The solid Cauchy transform ("$\overline{\partial}^{-1}$")

$$\mathcal{C}\left[f\right](z) = \int_{\mathbb{C}} \frac{1}{z-\zeta} f(\zeta)\, dA(\zeta)$$

Some Useful Facts (1 of 2)

- $H^{1,1}(\mathbb{C})$ is compactly embedded in $L^p(\mathbb{C})$ for $p \in (1, \infty)$
- If $p \in (2, \infty)$ and $f \in L^{2p/(p+2)}(\mathbb{C})$, then $Pf \in L^p(\mathbb{C})$ and

$$\left\| \mathcal{C}[f] \right\|_{L^p(\mathbb{C})} \leq C_p \left\| f \right\|_{L^{2p/(p+2)}(\mathbb{C})}$$

- If $f \in L^q(\mathbb{C}) \cap L^p(\mathbb{C})$ for $1 < q < 2 < p < \infty$, then $\mathcal{C}[f]$ is Hölder continuous of order $(p-2)/p$
- For the same p, q, the estimate

$$\left\| \mathcal{C}[f] \right\|_{C^0} \leq C_{p,q} \left(\left\| f \right\|_{L^p(\mathbb{C})} + \left\| f \right\|_{L^q(\mathbb{C})} \right)$$

holds.

Some Useful Facts (2 of 2)

Bukhgeim Integration by Parts Formula

Suppose S is a real-valued phase function and $f \in C_0^\infty(\mathbb{C})$ has support away from the critical points of S. Then

$$\int \frac{1}{z-\zeta} e^{i\omega S(\zeta)} f(\zeta)\, dA(\zeta) = \frac{e^{i\omega S(z)}}{i\omega (\overline{\partial} S)(z)} - \frac{1}{i\omega} \int \frac{1}{z-\zeta} e^{i\omega S(\zeta)} \overline{\partial} \left((\overline{\partial} S)^{-1}(\zeta) f(\zeta) \right) dA(\zeta)$$

The Defocussing Davey-Stewartson II Equation

The defocussing DS II equation describes the amplitude envelope $u = u(x, y, t)$ of a monochromatic, weakly nonlinear surface wave:

$$\begin{aligned} iu_t + 2\left(\partial^2 + \overline{\partial}^2\right) u + (g + \overline{g})u &= 0 \\ \overline{\partial} g + \partial\left(|u|^2\right) &= 0 \end{aligned}$$

The second equation determines g in terms of u at each time t. Another way of writing this equation is

$$iu_t + 2\left(\partial^2 + \overline{\partial}^2\right) u + 2\Re\left(\overline{\partial}^{-1}\partial|u|^2\right) u = 0$$

The DS II equation is a two-dimensional analogue of the defocussing cubic NLS in one dimension and is completely integrable.

We will outline how the inverse scattering method (ISM) may be used solve the Cauchy problem with initial data u_0 and prove that the defocussing DS II equation is globally well-posed in $H^{1,1}(\mathbb{C})$ One can also prove that the solution $u(x, y, t)$ is asymptotic in L^∞ to a solution of the linearized equation with initial data $v_0 = \mathcal{F}^{-1}\mathcal{R}u_0$

Key steps:

- Obtain good estimates and asymptotic behavior of scattering eigenfunctions
- Establish continuity properties of direct and inverse scattering maps between function spaces adapted to the problem

Remarks on the Literature

The $\overline{\partial}$-method for the DS II equations was pioneered by Fokas-Ablowitz and Beals-Coifman in the 1980's. The formulation we use here is closest to that of Sung (1994), who carried out a detailed study of the direct and inverse maps and the solution of DS II by inverse scattering, but did not consider the space $H^{1,1}(\mathbb{C})$. Brown (2000) proved multilinear estimates that are crucial to our approach, and used them to show that the direct and inverse scattering maps are Lipschitz continuous on L^2 for "small data."

ISM Schematics

To solve the DS II equation with initial data u_0 we will use the *direct scattering map* $\mathcal{R}$ to linearize the flow, and the *inverse scattering map* $\mathcal{I}$ to recover the solution:

$$
\begin{array}{ccc}
u_0(z) & \xrightarrow{\quad \mathcal{R} \quad (\textit{Direct}) \quad} & r_0 = \mathcal{R} u_0 \\
\text{DS II Flow} \Big\downarrow & & \Big\downarrow \text{Linearized Flow} \\
u(z,t) & \xleftarrow{\quad \mathcal{I} \quad (\textit{Inverse}) \quad} & r = e^{it(k^2 + \overline{k}^2)} r_0
\end{array}
$$

Lax Representation

The defocussing DS II admits a Lax representation $\dot{L} = [A, L]$ where

$$L = -\partial_x - iJ\partial_y + Q$$
$$A = B - Q\partial_y + iJ\partial_y^2$$

where

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & u \\ \overline{u} & 0 \end{pmatrix}$$

The equation $L\psi = 0$ defines a spectral problem, while the equation $\psi_t = A\psi$ implies a law of evolution for scattering data.

What the Lax Representation Tells You

If $\psi(t)$ solves

$$L(t)\psi(t) = 0$$

we can differentiate to obtain

$$\dot{L}(t)\psi + L(t)\dot{\psi} = 0.$$

Since $\dot{L}(t) = [A, L(t)]$ and $L(t)\psi(t) = 0$, we get

$$L(t)\left[\dot{\psi}(t) - A\psi(t)\right] = 0$$

If the function $\dot{\psi}(t) - A(t)\psi(t)$ satisfies the hypothesis of a vanishing theorem for $L(t)$, we get that $\dot{\psi}(t) = A(t)\psi(t)$ and hence an equation of motion for scattering data determined by asymptotics of ψ.

Spectral Problem (1 of 3)

We consider the spectral problem $L\psi = 0$ which may be written

$$\begin{pmatrix} \overline{\partial} & 0 \\ 0 & \partial \end{pmatrix} \psi = \frac{1}{2} \begin{pmatrix} 0 & u \\ \overline{u} & 0 \end{pmatrix} \psi$$

If $q = 0$, $z = x + iy$, and $k = k_1 + ik_2$, the function

$$\psi_0 = \begin{pmatrix} e^{ikz} & 0 \\ 0 & e^{-ik\overline{z}} \end{pmatrix}$$

is a solution for each $k \in \mathbb{C}$.

Spectral Problem (2 of 3)

$$\begin{pmatrix} \overline{\partial} & 0 \\ 0 & \partial \end{pmatrix} \psi = \frac{1}{2} \begin{pmatrix} 0 & u \\ \overline{u} & 0 \end{pmatrix} \psi$$

We seek solutions of the form

$$\psi = \psi_0 M$$

for a matrix-valued function M with

$$\lim_{|z| \to \infty} M(z,k) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

To convert to a pure $\overline{\partial}$-problem we'll make a change of variables

$$M = \begin{pmatrix} \mu_{11} & \mu_{12} \\ \overline{\mu_{21}} & \overline{\mu_{22}} \end{pmatrix}$$

Spectral Problem (3 of 3)

This leads to the spectral problem

$$\left(\overline{\partial}\mu\right)(z,k) = \frac{1}{2}e_{-k}(z)\begin{pmatrix} 0 & u(z) \\ u(z) & 0 \end{pmatrix}\overline{\mu(z,k)}$$

$$\lim_{|z|\to\infty}\mu(z,k) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where

$$e_k(z) = \exp i\left(kz + \overline{k}\overline{z}\right).$$

We'll show that this problem has a unique solution in $C^0(\mathbb{C})$ for each $k \in \mathbb{C}$, and obtain good large-k asymptotic estimates

Scattering Data (1 of 2)

Scattering data $r(k)$ and $s(k)$ are defined for $u \in C_0^\infty(\mathbb{C})$ by large-z behavior of solutions:

$$\begin{pmatrix} \mu_{11}(z,k) \\ \mu_{21}(z,k) \end{pmatrix} \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{z}\left[\frac{1}{2}\begin{pmatrix} s(k) \\ r(k) \end{pmatrix}\right] + \mathcal{O}\left(|z|^{-2}\right)$$

One also has integral representations

$$s(k) = \frac{1}{\pi}\int e_{-k}(z)u(z)\overline{\mu_{21}(z,k)}\,dA(z)$$
$$r(k) = \frac{1}{\pi}\int e_{-k}(z)u(z)\overline{\mu_{11}(z,k)}\,dA(z)$$

Scattering Data (2 of 2)

The $\mathcal{O}\left(|z|^{-1}\right)$ terms in the large-z expansion of μ_{11}, μ_{21} determine

$$s(k) = \frac{2}{\pi} \int e_{-k}(z) u(z) \overline{\mu_{21}(z,k)} \, dA(z)$$

$$r(k) = \frac{2}{\pi} \int e_{-k}(z) u(z) \overline{\mu_{11}(z,k)} \, dA(z)$$

Under the DS II flow:

- $s(k)$ is a conserved quantity
- $r(k)$ obeys a linear equation

Direct Scattering Map

The $\overline{\partial}$-problem

$$\left(\overline{\partial}_z \mu_{11}\right)(z,k) = \frac{1}{2} e_{-k}(z) u(z) \overline{\mu_{21}(z,k)}$$
$$\left(\overline{\partial}_z \mu_{21}\right)(z,k) = \frac{1}{2} e_{-k}(z) u(z) \overline{\mu_{11}(z,k)}$$

$$\lim_{|z|\to\infty} \left(\mu_{11}(z,k), \mu_{21}(z,k)\right) = (1,0)$$

and the representation formula

$$r(k) = \frac{1}{\pi} \int e_{-k}(z) u(z) \overline{\mu_{11}(z,k)}\, dA(z)$$

define the *direct scattering map* $\mathcal{R} : u \to r$, *provided* we can obtain good estimates on $\mu_{11}(z,k)$.

Integral Equations

Write (μ_1, μ_2) for (μ_{11}, μ_{21}) and Introduce the integral operator

$$T\psi = \frac{1}{2}\mathcal{C}\left[\left[e_{-k}u\overline{\psi}\right]\right.$$

where P is the solid Cauchy transform. Then the $\overline{\partial}$-system

$$\begin{aligned}\left(\overline{\partial}_z\mu_1\right)(z,k) &= \frac{1}{2}e_{-k}(z)u(z)\overline{\mu_2(z,k)}\\ \left(\overline{\partial}_z\mu_2\right)(z,k) &= \frac{1}{2}e_{-k}(z)u(z)\overline{\mu_1(z,k)}\end{aligned}$$

$$\lim_{|z|\to\infty}\left(\mu_1(z,k), \mu_2(z,k)\right) = (1,0)$$

is formally equivalent to

$$\begin{aligned}\mu_1 &= 1 + T(\mu_2)\\ \mu_2 &= T(\mu_1)\end{aligned}$$

$$T\psi = \frac{1}{2}\mathcal{C}\left[\left[e_{-k}u\overline{\psi}\right]\right.$$

$$\begin{aligned} \mu_1 &= 1 + T(\mu_2) \\ \mu_2 &= T(\mu_1) \end{aligned}$$

has the formal solution

$$\begin{aligned} \mu_1 &= (I - T^2)^{-1}1 \\ \mu_2 &= (I - T^2)^{-1}T1 \end{aligned}$$

This focusses attention on the resolvent $(I - T^2)^{-1}$ in $C^0(\mathbb{C})$

Three Little Facts

Recall that

$$T\psi = \frac{1}{2}\mathcal{C}\left[\left[e_{-k}u\overline{\psi}\right]\right]$$

so

$$T^2\psi = \frac{1}{4}\mathcal{C}\left[\left[e_{-k}u\overline{\mathcal{C}}\left[e_k\overline{u}\psi\right]\right]\right]$$

1. T is a compact antilinear operator on $C^0(\mathbb{C})$
2. $\ker(I - T^2)$ is trivial
3. $\|T^4\|_{\mathcal{B}(C^0)} \to 0$ as $|k| \to \infty$

These facts are (almost) enough to establish existence and good bounds on $(I - T^2)^{-1}$

Little Fact #1: T is compact (1 of 2)

Recall:

- A subset S of C^0 is precompact if the functions of S are uniformly equicontinuous and vanish at infinity at a uniform rate.
- A bounded operator $T : X \to X$ is compact if it maps bounded sets to precompact sets
- A norm-limit of compact operators is compact

Little Fact #1: T is compact (2 of 2)

Steps of Proof:

- $T := \frac{1}{2}\mathcal{C}\left[e_{-k}u\overline{(\,\cdot\,)}\right]$ maps C^0 into bounded, Hölder continuous functions
- $\|T\|_{\mathcal{B}(C^0)} \leq C\|u\|_{H^{1,1}}$
- If $u \in C_0^\infty(\mathbb{C})$, then $T\psi = \mathcal{O}\left(|z|^{-1}\right)$ uniformly in ψ in a bounded subset of C^0, so T is compact if $u \in C_0^\infty(\mathbb{C})$
- If $u_n \to u$ in $H^{1,1}$ and $u_n \in C_0^\infty(\mathbb{C})$, then $T_n \to T$ in $\mathcal{B}(C^0)$, so T is compact for any $u \in H^{1,1}(\mathbb{C})$

Little Fact #2: $\ker(I - T^2)$ is trivial (1 of 2)

$$T\psi = \mathcal{C}\left[e_k u \overline{\psi}\right]$$

1. If $T^2\psi = \psi$ then $\psi \in C^0 \cap L^p$ for all $p \in (2, \infty)$
2. $T^2\psi = \psi$ has a solution if and only if $(\mu_1, \mu_2) = (\psi, T\psi)$ solve

$$\overline{\partial}\mu_1 = \frac{1}{2} e_{-k} u \overline{\mu_2}$$
$$\overline{\partial}\mu_2 = e_{-k} u \overline{\mu_1}$$

3. Hence $\psi = \mu_1 + \mu_2$ satisfies

$$\overline{\partial}\psi = \frac{1}{2}(e_{-k} u)\overline{\psi}$$

Little Fact #2: $\ker(I - T^2)$ is Trivial (2 of 2)

$$\bar{\partial}\psi = \frac{1}{2}(e_{-k}u)\overline{\psi}$$

Theorem (Vekua, Brown-Uhlmann) *Suppose that*

$$\bar{\partial}v = av + b\overline{v}$$

where a, b belong to $L^2(\mathbb{C})$ and $v \in L^2_{\mathrm{loc}}(\mathbb{C}) \cap L^p(\mathbb{C})$ for some $p \in [1, \infty)$. Then $v = 0$.

- Conclude that $\psi = \mu_1 + \mu_2 = 0$ so $\mu_1 = -\mu_2$.
- Then apply the theorem again to conclude that $\mu_1 = \mu_2 = 0$.

Little Fact #3: $\| T \|_{\mathcal{B}(C^0)} \to 0$ as $|k| \to \infty$ (1 of 2)

$$(T\psi)(z) = \frac{1}{2\pi} \int_{\mathbb{C}} \frac{1}{z-\zeta} e_{-k}(\zeta) u(\zeta) \overline{\psi(\zeta)} \, dA(\zeta)$$

The Bukhgeim integration by parts formula with $S(\zeta) = k\zeta + \overline{k\zeta}$ implies that

$$\begin{aligned}(T\psi)(z) &= -\frac{e_{-k}(z)}{\pi i \overline{k}} u(z)\psi(z) \\ &\quad + \frac{1}{\pi i \overline{k}} \int \frac{e-k(\zeta)}{z-\zeta} \overline{\partial} \left(u(\,\cdot\,)\psi(\,\cdot\,) \right) \, dA(\zeta)\end{aligned}$$

so we get $\mathcal{O}\left(k^{-1}\right)$ decay in $L^p(\mathbb{C})$ at the cost of a derivative of ψ.

Little Fact #3: $\|T\|_{\mathcal{B}(C^0)} \to 0$ as $|k| \to \infty$ (2 of 2)

From integration by parts, we get

$$\|T\psi\|_{L^p(\mathbb{C})} \leq \frac{C}{|k|} \times \Big(\|u\|_{L^p(\mathbb{C})} \|\psi\|_{C^0(\mathbb{C})} + \left\|\overline{\partial} u\right\|_{L^2(\mathbb{C})} \|\psi\|_{L^p(\mathbb{C})} + \|u\|_{L^2(\mathbb{C})} \|\partial\psi\|_{L^p(\mathbb{C})} \Big)$$

By using smoothing properties of T, we can get

$$\left\|T^4\psi\right\|_{C^0(\mathbb{C})} \leq C(1+|k|)^{-1} \|\psi\|_{C^0(\mathbb{C})}.$$

The Resolvent (1 of 3)

From the "three little facts" we get:

Proposition *The resolvent* $(I - T^2)^{-1}$ *exists as a bounded operator from* $C^0(\mathbb{C})$ *to itself, with small norm if* $|k|$ *is large.*

However, we need to know that the resolvent is uniformly bounded in k and that the map

$$(k, u) \mapsto (I - T^2)^{-1}$$

is a bounded continuous map from $\mathbb{C} \times H^{1,1}(\mathbb{C})$ to $\mathcal{B}(C^0)$ because we need

$$(k, u) \mapsto \mu_1 := (I - T^2)^{-1} 1$$

to have analogous continuity properties.

The Resolvent (2 of 3)

Consider the map

$$(k, u) \mapsto (I - T^2)^{-1}$$

- We'll show that this map is continuous as a map from $\mathbb{C} \times X$ into $\mathcal{B}(C^0)$ for a Banach space X in which $H^{1,1}(\mathbb{C})$ is compactly embedded
- Using the fact that the continuous image of a compact set is compact, we'll show that the resolvent is uniformly bounded on bounded subsets of $\mathbb{C} \times H^{1,1}(\mathbb{C})$

Let $X = L^q(\mathbb{C}) \cap L^p(\mathbb{C})$ for p, q with $1 < q < 2 < p < \infty$

- $H^{1,1}(\mathbb{C})$ is compactly embedded in L^p for any $p \in (1, \infty)$
- The operator T satisfies a bound of the form

$$\|T\psi\|_{C^0(\mathbb{C})} \leq C \|u\|_X \|\psi\|_{C^0(\mathbb{C})} .$$

The Resolvent (3 of 3)

Proposition The map

$$(k, u) \mapsto (I - T^2)^{-1}$$

is continuous from $\mathbb{C} \times H^{1,1}(\mathbb{C})$ into $\mathcal{B}(C^0(\mathbb{C}))$ and uniformly bounded for $k \in \mathbb{C}$ and u in bounded subsets of $H^{1,1}(\mathbb{C})$.

By using the formula

$$(I - T^2)^{-1} = \sum_{j=0}^{N} T^{2j} + (I - T^2)^{-1} T^{2N+2}$$

we can obtain expansions modulo $\mathcal{O}\left(|k|^{-N}\right)$ in operator norm.

This implies that we obtain uniform bounds, continuity, and large-k expansions for the scattering solution $\mu_1(z, k)$.

Mapping Properties of the Direct Scattering Map

Recall that

$$r(k) = \frac{1}{\pi} \int_{\mathbb{C}} e_k(z) u(z) \overline{\mu_1(z,k)} \, dA(z).$$

Proposition The map $\mathcal{R}$ defined by $\mathcal{R}(u) = r$ is a Lipschitz continuous map from $H^{1,1}(\mathbb{C})$ to itself.

To prove the proposition, we need to show that the maps

$$\begin{aligned} u &\mapsto r(\,\cdot\,) \\ u &\mapsto (\,\cdot\,) r(\,\cdot\,) \\ u &\mapsto (\partial r)(\,\cdot\,) \\ u &\mapsto (\overline{\partial} r)(\,\cdot\,) \end{aligned}$$

are all continuous from $L^2(\mathbb{C})$ to itself.

Some Ideas of the Proof

- The expansion $\mu_1 = 1 + \sum_{j=1}^{N} (T^{2j}1) + (I - T^2)^{-1} T^{2N+2} 1$ implies that

$$r = \mathcal{F}u + \sum_{j=1}^{N} \int e_k u(T^{2j}1) + \mathcal{O}\left(|k|^{-N}\right)$$

- $\mathcal{F}$ is a Fourier transform that maps $H^{1,1}(\mathbb{C})$ to itself continuously
- The terms involving $T^{2j}1$ are explicit multilinear integrals that can be estimated by Brascamp-Lieb inequalities (Brown 2000)
- $(\cdot)r(\cdot)$ can be computed using integration by parts and the $\overline{\partial}$ equation solved by μ_1.
- ∂r and $\overline{\partial} r$ are related by the Beurling transform which is bounded from $L^p(\mathbb{C})$ to itself, $p \in (1, \infty)$.

The Inverse Scattering Map

The good news is that the inverse scattering map for DS II takes almost the same form as the direct scattering map, so all of our previous work carries over!

A "Dual" $\overline{\partial}$-Problem

Let (μ_1, μ_2) be as before for some $u \in H^{1,1}(\mathbb{C})$, let

$$r(k) = \frac{1}{\pi} \int e_k(z) u(z) \overline{\mu_1(z,k)} \, dA(z)$$

and let

$$(\nu_1, \nu_2) = (\mu_1, e_k \overline{\mu_2})$$

Proposition The functions (ν_1, ν_2) satisfy the $\overline{\partial}$-system

$$\left(\overline{\partial}_k \nu_1\right)(z,k) = -\frac{i}{2} e_k(z) \overline{r(k)} \, \overline{\nu_2(z,k)}$$
$$\left(\overline{\partial}_k \nu_2\right)(z,k) = -\frac{i}{2} e_k(z) \overline{r(k)} \, \overline{\nu_1(z,k)}$$

$$\lim_{|k| \to \infty} (\nu_1(z,k), \nu_2(z,k)) = (1, 0)$$

A Reconstruction Formula

Suppose that:

1. $u \in H^{1,1}(\mathbb{C})$ is given
2. (μ_1, μ_2) solve the $\overline{\partial}$-problem with $\overline{\partial}$-data u
3. (ν_1, ν_2) are defined by $(\nu_1, \nu_2) = (\mu_1, e_k \overline{\mu_2})$

Then

$$\begin{pmatrix} \nu_1(z,k) \\ \nu_2(z,k) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{z} \begin{pmatrix} -\frac{i}{4}\overline{\partial}^{-1}\left(|u|^2\right) \\ -\frac{i}{2}\overline{u} \end{pmatrix} + \mathcal{O}\left(|k|^{-2}\right)$$

so that

$$u(z) = \frac{1}{\pi} \int r(k) e_{-k}(z) \nu_1(z,k) \, dA(k)$$

The Inverse Scattering Map

The $\overline{\partial}$-problem

$$\left(\overline{\partial}_k \nu_1\right)(z,k) = \frac{1}{2} e_k(z) \overline{r(k) \nu_2(z,k)}$$
$$\left(\overline{\partial}_k \nu_2\right)(z,k) = \frac{1}{2} e_k(z) \overline{r(k) \nu_1(z,k)}$$

$$\lim_{|k|\to\infty} (\nu_1(z,k), \nu_2(z,k)) = (1,0)$$

together with the reconstruction formula

$$u(z) = \frac{1}{\pi} \int r(k) e_{-k}(z) \nu_1(z,k) \, dA(k)$$

define the *inverse scattering map* $\mathcal{I} : r \mapsto u$.

Propostion The inverse scattering map $\mathcal{I} : r \mapsto u$ is a Lipschitz continuous map from $H^{1,1}(\mathbb{C})$ to itself.

Proposition The maps $\mathcal{R}$ and $\mathcal{I}$ are mutual inverses on $H^{1,1}(\mathbb{C})$.

Time Evolution of Scattering Data

For the defocussing DS II equation with Schwarz class initial data, it can be shown that the scattering data $r(k,t)$ and $s(k,t)$ corresponding to $u(x,y,t)$ obey

$$\frac{d}{dt}r(k,t) = i\left(k^2 + \overline{k}^2\right) r(k,t)$$
$$\frac{d}{dt}s(k,t) = 0$$

so that, formally at least, we can solve DS II by

$$u(x,y,t) = \mathcal{I}\left[e^{it\left((\diamond)^2 + (\overline{\diamond})^2\right)}\left(\mathcal{R}u_0\right)(\diamond)\right](x,y)$$

Global Existence

The formula

$$u(x,y,t) = \mathcal{I}\left[e^{it\left((\diamond)^2+(\overline{\diamond})^2\right)}\left(\mathcal{R}u_0\right)(\diamond)\right](x,y)$$

defines a continuous map

$$\begin{aligned} \mathbb{R}\times H^{1,1}(\mathbb{C}) &\longrightarrow H^{1,1}(\mathbb{C}) \\ (t,u_0) &\mapsto u(\,\cdot\,,\,\cdot\,,t) \end{aligned}$$

by Lipschitz continuity of the maps $\mathcal{R}$ and $\mathcal{I}$

This gives "instant global well-posedness"!