NOTES ON THE SEMICLASSICAL LIMIT FOR THE DEFOCUSING DAVEY-STEWARTSON II EQUATION

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Abstract. This document summarizes and expounds upon some of the discussions held at BIRS in July-August 2012 on the topic of semiclassical analysis for the Davey-Stewartson II equation (and its spectral transforms). The final section includes a list of proposed numerical experiments related to the further development of the theory.

1. Introduction

By the semiclassical limit for the defocusing Davey-Stewartson II (DS-II) equation we mean the following Cauchy initial-value problem parametrized by $\epsilon > 0$:

$$\begin{aligned} i\epsilon q_t + 2\epsilon^2 \left(\overline{\partial}^2 + \partial^2\right) q + (g + \overline{g})\, q &= 0 \\ \overline{\partial} g + \partial\left(|q|^2\right) &= 0, \end{aligned} \tag{1}$$

for a complex-valued field $q = q^\epsilon(x, y, t)$ where

$$\partial := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \overline{\partial} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \tag{2}$$

subject to an initial condition of "oscillatory wavepacket" or "WKB" form:

$$q^\epsilon(x, y, 0) = A(x, y)e^{iS(x,y)/\epsilon}, \quad A > 0, \quad S \in \mathbb{R}. \tag{3}$$

Here A and S are functions independent of ϵ. The only explicit dependence on ϵ in this problem enters through the phase factor in the initial data and the coefficients of the DS-II equation. In general, the recipe for inserting ϵ's in Schrödinger-type equations to arrive at the semiclassical scaling is to replace the space-time gradient ∇ with $\epsilon\nabla$ and to divide the phase of the initial data by ϵ.

If we impose enough reasonable conditions on the functions A and S, then by the general theory described in [9] this problem has a unique global solution with nice properties for every $\epsilon > 0$, and the question we wish to address is *how does this well-defined solution $q = q^\epsilon(x, y, t)$ behave asymptotically as $\epsilon \downarrow 0$?* This problem is interesting because it sets up a competition between two space-time scales:

- On the one hand, the PDE seems to "prefer" solutions with rapid space-time variations: indeed setting $q^\epsilon(x, y, t) = Q(x/\epsilon, y/\epsilon, t/\epsilon)$, one sees that Q satisfies (1) with $\epsilon = 1$.
- On the other hand, the simple scaling that removes the ϵ from the PDE has the side effect of stretching the initial data for Q in the (x, y)-directions by a factor of ϵ^{-1}. This makes the norm of the initial data for Q very large as $\epsilon \downarrow 0$.

If one resists the temptation to scale the ϵ out of the PDE at the cost of stretching out the initial data, then one sees that the "natural" space-time scales of the PDE are $\Delta x \sim \Delta y \sim \Delta t = O(\epsilon)$ while those of the functions A and S in the initial data are $\Delta x \sim \Delta y \sim \Delta t = O(1)$. One therefore expects that the solution $q^\epsilon(x, y, t)$ will exhibit a multiscale structure when ϵ is small.

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2. Madelung's Quantum Fluid Dynamics

We can eliminate the imaginary part of g from (1) and write the DS-II equation in the form

$$
\begin{aligned}
i\epsilon q_t + \frac{\epsilon^2}{2}\left(q_{xx} - q_{yy}\right) + 2Mq &= 0 \\
M_{xx} + M_{yy} &= \left(|q|^2\right)_{yy} - \left(|q|^2\right)_{xx}.
\end{aligned} \tag{4}
$$

Here $M = M^\epsilon(x, y, t)$ is just the real part of g. Note that when written in this form it is clear that for q independent of y the DS-II equation reduces to the defocusing nonlinear Schrödinger equation

$$
i\epsilon q_t + \frac{\epsilon^2}{2} q_{xx} - 2|q|^2 q = 0 \tag{5}
$$

while for q independent of x the DS-II equation reduces to the *time-reversed, or complex conjugated* defocusing nonlinear Schrödinger equation

$$
-i\epsilon q_t + \frac{\epsilon^2}{2} q_{yy} - 2|q|^2 q = 0. \tag{6}
$$

In the early days of quantum mechanics, Erwin Madelung had the idea [7] that Schrödinger's equation could be interpreted as describing the dynamics of a kind of compressible fluid whose density and velocity field satisfied familiar Euler-type equations supplemented with "quantum corrections". His idea allows us to guess what the semiclassical limit of the DS-II equation might be like.

Let us assume only that $|q| > 0$ for all (x, y, t), and unambiguously represent q in the form (resembling the initial data):

$$
q^\epsilon(x, y, t) = A^\epsilon(x, y, t)e^{iS^\epsilon(x,y,t)/\epsilon}. \tag{7}
$$

Inserting this form into (4), dividing out the common phase factor from the first equation and separating it into real and imaginary parts gives, without approximation, the following system governing the three real-valued fields $A = A^\epsilon(x, y, t)$, $S = S^\epsilon(x, y, t)$, and $M = M^\epsilon(x, y, t)$:

$$
\begin{aligned}
S_t + \frac{1}{2}S_x^2 - \frac{1}{2}S_y^2 - 2M &= \frac{\epsilon^2}{2}\frac{A_{yy} - A_{xx}}{A} \\
A_t + A_x S_x + \frac{1}{2}AS_{xx} - A_y S_y - \frac{1}{2}AS_{yy} &= 0 \\
M_{xx} + M_{yy} + \left(A^2\right)_{xx} - \left(A^2\right)_{yy} &= 0.
\end{aligned} \tag{8}
$$

This system is to be solved with the ϵ-independent initial data $A^\epsilon(x, y, 0) = A(x, y)$ and $S^\epsilon(x, y, 0) = S(x, y)$. This situation obviously invites the neglect of the formally small terms proportional to ϵ^2 on the right-hand side of (8). The resulting system is called the *dispersionless DS-II system.*

Madelung's quantum fluid density is by definition the function

$$
\rho = \rho^\epsilon(x, y, t) := A^\epsilon(x, y, t)^2 \tag{9}
$$

and his quantum fluid velocity field is (here and below ∇ is the gradient in just the spatial variables (x, y))

$$
\mathbf{u} = \mathbf{u}^\epsilon(x, y, t) := \nabla S^\epsilon(x, y, t). \tag{10}
$$

The dispersionless DS-II system can be written in terms of the density and velocity fields as follows:

$$
\begin{aligned}
\mathbf{u}_t + \frac{1}{2}\nabla\left(\mathbf{u}\cdot\sigma_3\mathbf{u}\right) - 2\nabla M &= \mathbf{0} \\
\rho_t + \operatorname{div}\left(\rho\sigma_3\nabla\mathbf{u}\right) &= 0 \\
\Delta M + \operatorname{div}\left(\sigma_3\nabla\rho\right) &= 0.
\end{aligned} \tag{11}
$$

Here $\sigma_3 = \operatorname{diag}(1, -1)$ is a Pauli matrix. We see that M has the interpretation of a kind of fluid pressure. In fact, if we were to replace σ_3 by the identity matrix, these would be the equations of motion for a physical compressible fluid.

From the mathematical point of view, we might guess that when $\epsilon > 0$ is small, the solution $q^\epsilon(x, y, t)$ of the semiclassical DS-II equation might be accurately modeled by the solution of the Cauchy initial-value problem for (11) with initial data $\rho(x,y,0)=A(x,y)^2$ and $\mathbf{u}(x,y,0)=\nabla S(x,y)$. *So one question for the PDE experts that arises is whether the latter problem has a local solution in any reasonable (strong) sense.*

If the dispersionless problem is locally well-posed, then the logical conjecture is that the solution of (1) can be approximated for small ϵ by the dispersionless limit over some finite time interval independent of ϵ. In the case of the defocusing nonlinear Schrödinger equation in $1+1$ dimensions, the Madelung dispersionless system corresponding to (11) is a hyperbolic quasilinear system, and local well-posedness is guaranteed; the logical conjecture has been proven in this case by several different methods ranging from energy estimates applied to a Madelung-type ansatz [2] to Lax-Levermore variational theory [3]. No doubt matrix steepest-descent type Riemann-Hilbert techniques could also be applied to this problem following virtually exactly the same kind of analysis as was developed for the small-dispersion limit of the Korteweg-de Vries equation [1] although to my knowledge it has not been written up by anyone (probably it is considered fruit that is hanging too low to be interesting).

Even if one has local well-posedness for the dispersionless system (11), one does not expect (based on the known examples in $1+1$ dimensions anyway) to have global well-posedness. One expects instead that the solution of the dispersionless system develops singularities (shocks, gradient catastrophes, or caustics) in finite time. While it might seem attractive to pass to an appropriate notion of weak solution if this occurs, experience has shown that what the breakdown signals is that the terms proportional to ϵ^2 that were neglected in deriving (11) can no longer be discarded and must instead be included at the same order. In other words, the DS-II PDE responds to shock formation with the generation of small-scale (order ϵ) structures in the functions $A^\epsilon(x,y,t)$, $S^\epsilon(x,y,t)$, and $M^\epsilon(x,y,t)$. Once these structures form locally near the shock point, a different kind of ansatz is required locally for q, and a more complicated system obtained by Whitham averaging would be expected to take the place of (11).

3. Inverse Scattering Transform

We follow the notation of [9] (some signs are different on the Drupal website). Some numerical factors in the below formulae may differ from what one reads in earlier versions of [9] but in recent discussions with Peter we have agreed that these constants are correct as rendered below. The key idea here is to rewrite the formulas for the scattering maps and time dependence with the appropriate ϵ scalings.

3.1. Direct transform.

Consider the system of linear equations

$$\begin{aligned} \epsilon\overline{\partial}\psi_1 &= \tfrac{1}{2}q\psi_2 \\ \epsilon\partial\psi_2 &= \tfrac{1}{2}\overline{q}\psi_1 \end{aligned} \tag{12}$$

to which we seek for each fixed time t the unique (complex geometrical optics) solution $\psi_j=\psi_j^\epsilon(x,y;\kappa,\sigma,t)$ parametrized by the additional complex parameter $k=\kappa+i\sigma\in\mathbb{C}$ that satisfies the asymptotic conditions:

$$\begin{aligned} &\lim_{|z|\to\infty}\psi_1^\epsilon(x,y;\kappa,\sigma,t)e^{-kz/\epsilon}=1 \\ &\lim_{|z|\to\infty}\psi_2^\epsilon(x,y;\kappa,\sigma,t)e^{-\overline{k}\overline{z}/\epsilon}=0, \end{aligned} \tag{13}$$

where $z=x+iy$. The *reflection coefficient* $r=r^\epsilon(\kappa,\sigma;t)$ is defined in terms of $\psi_2^\epsilon(x,y;\kappa,\sigma,t)$ as follows:

$$e^{-kz/\epsilon}\overline{\psi_2^\epsilon(x,y;\kappa,\sigma,t)}=\tfrac{1}{2}r^\epsilon(\kappa,\sigma;t)z^{-1}+O(|z|^{-2}),\quad |z|\to\infty. \tag{14}$$

3.2. Time dependence.

As q evolves in time t according to (1), the reflection coefficient evolves by a trivial phase factor:

$$r^\epsilon(\kappa,\sigma;t)=r_0^\epsilon(\kappa,\sigma)e^{4it\Re\{k^2\}/\epsilon},\quad k=\kappa+i\sigma,\quad r_0^\epsilon(\kappa,\sigma):=r^\epsilon(\kappa,\sigma;0). \tag{15}$$

For convenience we define

$$R^\epsilon(\kappa,\sigma;x,y,t):=r^\epsilon(\kappa,\sigma;t)e^{2i\Im\{kz\}/\epsilon},\quad z=x+iy. \tag{16}$$

3.3. **Inverse transform.** I have derived the inverse-scattering problem from scratch following the arguments in section 4 of [9] to make sure that the ϵ comes in correctly. I obtained the following. Consider the system of linear equations

$$\begin{aligned} \epsilon\overline{\partial}_k\nu_1 &= \tfrac{1}{2}\overline{R^\epsilon(\kappa,\sigma;x,y,t)}\overline{\nu}_2 \\ \epsilon\overline{\partial}_k\nu_2 &= \tfrac{1}{2}\overline{R^\epsilon(\kappa,\sigma;x,y,t)}\overline{\nu}_1 \end{aligned} \tag{17}$$

where

$$\overline{\partial}_k := \frac{1}{2}\left(\frac{\partial}{\partial\kappa} + i\frac{\partial}{\partial\sigma}\right) \tag{18}$$

and where the solution $\nu_j = \nu_j^\epsilon(\kappa,\sigma;x,y,t)$ is sought that satisfies

$$\lim_{|k|\to\infty}\nu_1^\epsilon(\kappa,\sigma;x,y,t) = 1 \quad\text{and}\quad \lim_{|k|\to\infty}\nu_2^\epsilon(\kappa,\sigma;x,y,t) = 0. \tag{19}$$

Using the identities

$$\nu_1 = \mu_1 = e^{-kz/\epsilon}\psi_1 \quad\text{and}\quad \nu_2 = e^{(\overline{kz}-kz)/\epsilon}\overline{\mu}_2 = e^{-kz/\epsilon}\psi_2 \tag{20}$$

and the complex conjugate of the second equation of the system (12) gives the formula

$$q^\epsilon(x,y,t) = 2\epsilon\overline{\left[\frac{\partial\psi_2}{\psi_1}\right]} = 2\epsilon\frac{\overline{\partial}\,\overline{\psi}_2}{\overline{\psi}_1} = 2\frac{\overline{k}\,\overline{\nu}_2 + \epsilon\overline{\partial}\,\overline{\nu}_2}{\overline{\nu}_1}. \tag{21}$$

The right-hand side is in fact independent of $k = \kappa + i\sigma$, so we can let $|k|\to\infty$ and use the asymptotics for ν_j with respect to k to get the reconstruction formula

$$q^\epsilon(x,y,t) = 2\lim_{|k|\to\infty}\overline{k}\,\overline{\nu_2^\epsilon(\kappa,\sigma;x,y,t)} \tag{22}$$

4. WKB Method for Calculating the Reflection Coefficient

If the initial data is given in the form (3), then (12) takes the form of a linear system of partial differential equations with highly oscillatory coefficients:

$$\begin{aligned} \epsilon\overline{\partial}\psi_1 &= A(x,y)e^{iS(x,y)/\epsilon}\psi_2 \\ \epsilon\partial\psi_2 &= A(x,y)e^{-iS(x,y)/\epsilon}\psi_1. \end{aligned} \tag{23}$$

Let us assume for simplicity that $A(x,y)$ is a strictly positive Schwartz-class function, and that the real-valued phase is asymptotically linear: $S(x,y) = wz + \overline{wz} + O(1)$ as $z\to\infty$ for some $w\in\mathbb{C}$, in the sense that

$$\partial S(x,y)\to w \quad\text{and}\quad \overline{\partial}S(x,y)\to\overline{w}, \quad z\to\infty. \tag{24}$$

The oscillatory factors $e^{\pm iS(x,y)/\epsilon}$ can be removed by the substitution

$$\begin{bmatrix}\psi_1\\ \psi_2\end{bmatrix} = e^{iS(x,y)\sigma_3/(2\epsilon)}\begin{bmatrix}\chi_1\\ \chi_2\end{bmatrix} \tag{25}$$

leading to the equivalent system

$$\epsilon\begin{bmatrix}\overline{\partial} & 0\\ 0 & \partial\end{bmatrix}\begin{bmatrix}\chi_1\\ \chi_2\end{bmatrix} = \begin{bmatrix}-i\frac{1}{2}\overline{\partial}S & A\\ A & i\frac{1}{2}\partial S\end{bmatrix}\begin{bmatrix}\chi_1\\ \chi_2\end{bmatrix}. \tag{26}$$

We would like to develop a perturbation scheme based on the limit $\epsilon\to 0$. As the system is currently written, the obvious thing to do would be to neglect the derivatives of χ_j, but this is only consistent with a nontrivial solution if the coefficient matrix on the right-hand side is singular, which can be assumed to be a non-generic (with respect to $(x,y)\in\mathbb{R}^2$) phenomenon.

The way around this difficulty is to introduce an ϵ-independent complex scalar field $f:\mathbb{R}^2\to\mathbb{C}$ and make an exponential gauge transformation of the form

$$\chi_j = e^{f(x,y)/\epsilon}\phi_j, \quad j = 1,2. \tag{27}$$

This transforms (26) into the form

$$\epsilon \begin{bmatrix} \overline{\partial} & 0 \\ 0 & \partial \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \mathbf{M}(x,y) \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}, \tag{28}$$

where $\mathbf{M}(x,y)$ is the ϵ-independent matrix

$$\mathbf{M}(x,y) := \begin{bmatrix} -i\frac{1}{2}\overline{\partial}S - \overline{\partial}f & A \\ A & i\frac{1}{2}\partial S - \partial f \end{bmatrix}. \tag{29}$$

Now there are three scalar unknowns, ϕ_1, ϕ_2, and f. However we may now choose to determine f such that the augmented coefficient matrix $\mathbf{M}(x,y)$ on the right-hand side of (28) is singular for all $(x,y) \in \mathbb{R}^2$. This condition is obviously equivalent to the *eikonal equation* for f:

$$\det(\mathbf{M}(x,y)) = 0 \quad \Leftrightarrow \quad \left[\overline{\partial}f + i\tfrac{1}{2}\overline{\partial}S\right]\left[\partial f - i\tfrac{1}{2}\partial S\right] = A^2. \tag{30}$$

We should keep in mind the asymptotic normalization (13) of the functions ψ_j as $z \to \infty$, and see what conditions this places on ϕ_j and f. Obviously (13) implies

$$\begin{aligned} \lim_{|z|\to\infty} \phi_1 e^{f/\epsilon} e^{iS/(2\epsilon)} e^{-kz/\epsilon} &= 1 \\ \lim_{|z|\to\infty} \phi_2 e^{f/\epsilon} e^{-iS/(2\epsilon)} e^{-\overline{k}\overline{z}/\epsilon} &= 0 \end{aligned} \tag{31}$$

Since S is real, and since the second limit is zero, these two conditions can be combined to read

$$\lim_{|z|\to\infty} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \exp\left(\frac{1}{\epsilon}\left[f + i\frac{1}{2}S - kz\right]\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{32}$$

Since we want to be able to accurately represent ϕ_j using asymptotic power series in ϵ, in particular we want ϕ_j to have simple asymptotics as $z \to \infty$, so we now impose on the WKB exponent function f the normalization condition:

$$f(x,y) = kz - i\frac{1}{2}S(x,y) + o(1), \quad z = x + iy \to \infty. \tag{33}$$

Under this condition, (32) becomes simply

$$\lim_{|z|\to\infty} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{34}$$

It is an interesting question for the PDE experts as to whether the eikonal equation (30) *has a smooth solution for some/all* $k \in \mathbb{C}$ *under the normalization condition* (33)*, and if so, whether that solution is unique.*

Given a nice solution $f(x,y)$ of the eikonal equation (30) satisfying also the normalization condition (33), one attempts to find ϕ_j in the form of asymptotic power series in ϵ:

$$\begin{bmatrix} \phi_1^\epsilon(x,y) \\ \phi_2^\epsilon(x,y) \end{bmatrix} \sim \sum_{n=0}^{\infty} \Phi^{(n)}(x,y)\epsilon^n, \quad \epsilon \to 0, \quad \Phi^{(n)}(x,y) = \begin{bmatrix} \Phi_1^{(n)}(x,y) \\ \Phi_2^{(n)}(x,y) \end{bmatrix}. \tag{35}$$

Substituting into (28) and matching the terms with the same powers of ϵ one finds firstly that $\Phi^{(0)}(x,y) \in \ker(\mathbf{M}(x,y))$ (of course the kernel is nonempty by the choice of f), which determines $\Phi^{(0)}(x,y)$ up to a scalar multiple (which can depend on $(x,y) \in \mathbb{R}^2$). Then from the higher-order terms one obtains the recurrence relations:

$$\mathbf{M}(x,y)\Phi^{(n)}(x,y) = \begin{bmatrix} \overline{\partial} & 0 \\ 0 & \partial \end{bmatrix} \Phi^{(n-1)}(x,y). \tag{36}$$

As $\mathbf{M}(x,y)$ is singular, at each order there is a solvability condition to ensure that the right-hand side is in $\operatorname{ran}(\mathbf{M}(x,y))$, which determines the free scalar multiple of the kernel of $\mathbf{M}(x,y)$ that can be included for free in the solution from the previous order.

Obviously, we have

$$\ker(\mathbf{M}(x,y)) = \operatorname*{span}_{\mathbb{C}(x,y)} \begin{bmatrix} A(x,y) \\ \overline{\partial}f(x,y) + i\frac{1}{2}\overline{\partial}S(x,y) \end{bmatrix} = \operatorname*{span}_{\mathbb{C}(x,y)} \begin{bmatrix} \partial f(x,y) - i\frac{1}{2}\partial S(x,y) \\ A(x,y) \end{bmatrix}. \tag{37}$$

Keeping in mind that $A(x,y) = |q(x,y,0)| \to 0$ as $z \to \infty$, let us write $\Phi^{(0)}(x,y)$ in the form

$$\Phi^{(0)}(x,y) = \frac{\alpha(x,y)}{k-iw}\begin{bmatrix}\partial f(x,y) - i\frac{1}{2}\partial S(x,y) \\ A(x,y)\end{bmatrix} \tag{38}$$

(note that this form suggests some sort of singularity when $k = iw$) for a scalar field $\alpha(x,y)$, which according to (24), (33), and (34) should satisfy $\alpha(x,y) \to 1$ as $z \to \infty$. Then it follows that

$$\begin{bmatrix}\overline{\partial} & 0 \\ 0 & \partial\end{bmatrix}\Phi^{(0)}(x,y) = \frac{1}{k-iw}\begin{bmatrix}\overline{\partial}\alpha(x,y)[\partial f(x,y) - i\frac{1}{2}\partial S(x,y)] + \alpha(x,y)[\partial\overline{\partial} f(x,y) - i\frac{1}{2}\partial\overline{\partial}S(x,y)] \\ \partial\alpha(x,y)\,A(x,y) + \alpha(x,y)\partial A(x,y)\end{bmatrix}. \tag{39}$$

Then, since

$$\mathrm{ran}(\mathbf{M}(x,y)) = \operatorname*{span}_{\mathbb{C}(x,y)}\begin{bmatrix}-i\frac{1}{2}\overline{\partial}S(x,y) - \overline{\partial}f(x,y) \\ A(x,y)\end{bmatrix} = \operatorname*{span}_{\mathbb{C}(x,y)}\begin{bmatrix}A(x,y) \\ i\frac{1}{2}\partial S(x,y) - \partial f(x,y)\end{bmatrix}, \tag{40}$$

the solvability condition for $\Phi^{(1)}(x,y)$ can be written as

$$\det\left(\begin{bmatrix}-i\frac{1}{2}\overline{\partial}S(x,y) - \overline{\partial}f(x,y) \\ A(x,y)\end{bmatrix}, \begin{bmatrix}\overline{\partial} & 0 \\ 0 & \partial\end{bmatrix}\Phi^{(0)}(x,y)\right) = 0, \tag{41}$$

which is easily seen to be a scalar linear equation for $\alpha(x,y)$, and this together with the asymptotic condition $\alpha(x,y) \to 0$ as $z \to \infty$ should determine $\Phi^{(0)}(x,y)$ completely. With $\Phi^{(0)}(x,y)$ so-determined, the subsequent corrections should be sought subject to the normalization condition $\Phi^{(n)}(x,y) \to \mathbf{0}$ as $z \to \infty$ since the leading term of the series (35) has already satisfied (34).

Assuming that all of this goes through (being able to solve the nonlinear eikonal equation globally for $f(x,y)$, being able to construct $\alpha(x,y)$, and being able to prove the accuracy of the WKB approximation, namely that $\phi_j^\epsilon(x,y) - \Phi_j^{(0)}(x,y) \to 0$ in a suitable sense as $\epsilon \to 0$), we get an approximate formula for the initial reflection coefficient $r_0^\epsilon(\kappa,\sigma)$. Indeed, in terms of ϕ_2 the exact formula for $r_0^\epsilon(\kappa,\sigma)$ (recall $k = \kappa + i\sigma$) can be written as (using (14), (25), (27), and (33))

$$r_0^\epsilon(\kappa,\sigma) = 2\lim_{z\to\infty} ze^{-2i\Im\{kz\}/\epsilon}\overline{\phi_2^\epsilon(x,y)}. \tag{42}$$

Replacing ϕ_2 with its leading-order approximation $\Phi_2^{(0)}(x,y) = \alpha(x,y)A(x,y)/(k-iw)$ yields the approximate formula

$$r_0^\epsilon(\kappa,\sigma) \approx \frac{2}{\overline{k}+i\overline{w}}\lim_{z\to\infty} ze^{-2i\Im\{kz\}/\epsilon}\overline{\alpha(x,y)}A(x,y) = 0 \tag{43}$$

(the limit is zero because $\alpha \to 1$ and A is Schwartz-class).

The fact that the above limit is zero should probably be interpreted as letting us know that something necessarily goes wrong, at least for some $k \in \mathbb{C}$, with the various steps leading to this result. The most obvious culprit is the claim that for all $k \in \mathbb{C}$ there exists a global smooth solution of the eikonal equation (30) subject to the asymptotic condition (33). In fact, even in the case of the corresponding WKB problem for the $1+1$ defocusing nonlinear Schrödinger equation (a linear ODE takes the place of (12) and the analogue of the eikonal equation is a real first-order scalar nonlinear ODE), there exist values of the spectral parameter where the WKB method fails due to the presence of *turning points* in the physical space at locations parametrized by k. Local analysis near the turning points allows one to connect solutions through these bad spots and the result is an approximation for the reflection coefficient that does not vanish to leading order (for those k for which there exist turning points, that is). So, we should try to understand how the eikonal equation (30) breaks down, and we should try to figure out what the 2-dimensional analogue of turning points are!

4.1. **First simplification:** $S(x,y) \equiv 0$**.** In the nontrivial case that the phase S vanishes identically, this equation simplifies:

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = 4A(x,y)^2, \quad \text{if } S(x,y) \equiv 0. \tag{44}$$

One can surely assume for simplicity that $A(x,y)$ is a real-valued non-vanishing Schwartz-class function. The normalization condition (33) shows that even in the special case that $S(x,y) \equiv 0$, the WKB exponent f is essentially complex-valued for $k \neq 0$.

When $k = 0$, one has (44) subject to the condition $f \to 0$ as $z \to \infty$. One can solve this problem using the method of characteristics, assuming (consistently, in this case) that f is real. The characteristic equations are:

$$\frac{df}{ds} = \mathbf{p}\cdot\mathbf{p}, \quad \frac{d\mathbf{x}}{ds} = \mathbf{p}, \quad \frac{d\mathbf{p}}{ds} = 2\nabla A(x,y)^2, \quad \mathbf{p} = \begin{bmatrix} p \\ q \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}. \tag{45}$$

In particular, the $(\mathbf{x}, \mathbf{p})$ equations are Hamilton's equations describing a unit mass particle moving in $\mathbb{R}^2$ at position $\mathbf{x}$ with momentum $\mathbf{p}$, with potential energy given by $V(\mathbf{x}) = -2A(x,y)^2$. The condition $f \to 0$ as $\mathbf{x} \to \infty$ means that the particle starts at infinity with zero momentum. Since one can go to infinity in any direction, one should imagine an ensemble of particles all moving independently and initially distributed uniformly on a circle of large radius, with zero initial momentum. These particles will then begin to fall into the potential well $V(\mathbf{x})$ formed by the negative of the square of the amplitude function. If A is such that particles from the ensemble collide with one another at some point (x,y), this means that a singularity will form in $f(x,y)$ at such a point.

For $k \neq 0$, the asymptotic condition (33) requires that f be sought as a complex-valued function. In this case, it is easier to rewrite (44) in the complex form

$$\overline{\partial} f(x,y) \partial f(x,y) = A(x,y)^2. \tag{46}$$

Introducing two fields $G := \overline{\partial} f$ and $H := \partial f$, the eikonal equation becomes the algebraic equation $GH = A^2$ and instead we obtain the first-order PDE relating G and H:

$$\partial G - \overline{\partial} H = 0. \tag{47}$$

Eliminating G by $G = A^2/H$ yields a complex scalar quasilinear equation for H:

$$A^2 \partial H + H^2 \overline{\partial} H = H \partial(A^2). \tag{48}$$

To determine the type of this equation, we can introduce the real and imaginary parts of H by $H = \zeta + i\xi$ and obtain the system

$$\mathbf{X}\frac{\partial}{\partial x}\begin{bmatrix} \zeta \\ \xi \end{bmatrix} + \mathbf{Y}\frac{\partial}{\partial y}\begin{bmatrix} \zeta \\ \xi \end{bmatrix} = \begin{bmatrix} (A^2)_x & (A^2)_y \\ -(A^2)_y & (A^2)_x \end{bmatrix}\begin{bmatrix} \zeta \\ \xi \end{bmatrix}, \tag{49}$$

where the coefficient matrices are

$$\mathbf{X} := \begin{bmatrix} A^2 + \zeta^2 - \xi^2 & -2\zeta\xi \\ 2\zeta\xi & A^2 + \zeta^2 - \xi^2 \end{bmatrix}, \quad \mathbf{Y} := \begin{bmatrix} -2\zeta\xi & A^2 - \zeta^2 + \xi^2 \\ -A^2 + \zeta^2 - \xi^2 & -2\zeta\xi \end{bmatrix}. \tag{50}$$

For such a system, the characteristic velocities dy/dx are by definition the eigenvalues of the matrix $\mathbf{X}^{-1}\mathbf{Y}$, and the system is of hyperbolic type if the eigenvalues are real and distinct, and of elliptic type otherwise. Here, the matrix $\mathbf{X}^{-1}\mathbf{Y}$ is obviously real symmetric:

$$\mathbf{X}^{-1}\mathbf{Y} = \frac{1}{(A^2 + \zeta^2 - \xi^2)^2 + 4\zeta^2\xi^2}\begin{bmatrix} -4A^2\zeta\xi & A^4 - (\zeta^2 + \xi^2)^2 \\ A^4 - (\zeta^2 + \xi^2)^2 & -4A^2\zeta\xi \end{bmatrix} \tag{51}$$

and hence this is indeed a hyperbolic system, with characteristic velocities given explicitly by

$$\frac{dy}{dx} = \frac{-4A^2\zeta\xi \pm (A^4 - (\zeta^2 + \xi^2))}{(A^2 + \zeta^2 - \xi^2)^2 + 4\zeta^2\xi^2}. \tag{52}$$

Note that this is apparently a different result from what was reported during the Banff meeting, where I claimed this system to be of nonlinear elliptic type. As it is a system instead of a scalar equation (as was the case for $k = 0$), it generally cannot be integrated by reducing to a system of ODEs along one family of characteristics (there are two families of characteristics with distinct speeds). Nonetheless, one expects similar phenomena as in the $k = 0$ case: a global solution is very unlikely due to shock formation as one moves in from $z = \infty$.

That singularities are to be expected even for smooth $A(x,y)$ can be seen from some simple calculations too. For example, note that the function

$$f(x,y) = \log(z) + |z|^2, \tag{53}$$

which has a branch point singularity at $z = 0$, satisfies (46) in the case that $A(x,y)^2 = 1 + |z|^2$.

4.2. **Second simplification: $S(x,y)\equiv 0$ and $A(x,y)$ radially symmetric.** When $A(x,y)$ is radially symmetric, the problem is quite easy in the case that $k=0$, since the gradient of $A(x,y)^2$ points toward the origin, and consequently all of the particles move radially along characteristics toward the origin, and they avoid collision until they arrive (simultaneously!) at that point. Of course this just means that one should seek f as a function of $r=\sqrt{x^2+y^2}$ alone. The equation (44) then becomes simply

$$f'(r)^2 = 4A(r)^2 \quad \text{with solutions} \quad f(r) = \pm 2\int_r^\infty A(r')\,dr'. \tag{54}$$

We therefore obtain two distinct solutions that satisfy the boundary condition $f\to 0$ as $z\to\infty$. Note however, that neither of them is smooth at the origin unless $A(0)=0$, a fact that reflects the collision-of-particles argument as well.

5. Proposed Numerical Experiments

We clearly need some hints in order to figure out what to try next. So we turn to numerical experimentation. Here are several numerical experiments to try. The following problems are all related to the direct scattering problem.

(1) Suppose that $A(x,y)=e^{-(x^2+y^2)}$ and $S(x,y)\equiv 0$. This means that we are interested in the solution of the Davey-Stewartson equation (1) with purely real and positive radially symmetric Gaussian initial data independent of ϵ: $q(x,y,0)=e^{-(x^2+y^2)}$. By numerically solving the system (12) subject to the conditions (13) for a grid of values of $k=\kappa+i\sigma\in\mathbb{C}$, determine the reflection coefficient $r_0^\epsilon(\kappa,\sigma)$ on this grid by means of (14). This numerical experiment requires the choice of a value of $\epsilon>0$, and the way to see what to expect in the limit is to compare several different values. I suggest trying this for:
   (a) $\epsilon=2$.
   (b) $\epsilon=1$.
   (c) $\epsilon=0.5$.
   (d) $\epsilon=0.25$.

   It is to be expected that the smaller ϵ becomes, the more numerically challenging the problem should be. In particular one expects structures to appear in the reflection coefficient on short k-scales proportional to ϵ, so in addition to needing more points or Fourier modes in the z-plane to resolve the reflection coefficient for a single value of k, one needs to refine the grid in the k-plane as well to resolve the structure of r_0^ϵ. It would be useful to see plots of the real and imaginary parts of $r_0^\epsilon(\kappa,\sigma)$ as functions of the two coordinates in the k-plane.

(2) Assuming that experiment goes well, I would suggest trying the same experiment but breaking radial symmetry while maintaining $S(x,y)\equiv 0$. Try the same values of ϵ but take $A(x,y)=e^{-(x^2+\frac{1}{4}y^2)}$, say.

(3) Next, one should try to see what happens if a phase is included. Try $A(x,y)=e^{-(x^2+y^2)}$ and also $S(x,y)=e^{-(x^2+y^2)}$. Note that in this case the initial condition $q(x,y,0)$ for the DS-II equation has radial symmetry, but it is no longer independent of ϵ. The same four values of ϵ as before could be tried.

(4) The Gaussian phase function from the previous experiment tends to zero rapidly as $z\to\infty$, but it would also be interesting to see what happens if S is asymptotically linear, say $S(x,y)=wz+\overline{wz}+e^{-(x^2+y^2)}$ for some choice of a nonzero complex number w, say $w=1$. The formal WKB theory suggests that a problem occurs in the k-plane at the point $k=iw$. Can numerics help to explain what is going on near this point? I'd suggest keeping the Gaussian amplitude function $A(x,y)=e^{-(x^2+y^2)}$ and trying the same four values of ϵ.

(5) Although it is really beyond the scope of these notes, I can't help mentioning the following as a possible problem for numerical exploration. Suppose we consider instead the *focusing* version of the DS-II equation. The direct scattering problem is the same (see (12)–(13)) except that in the second equation one replaces $\overline{q}$ with $-\overline{q}$. Recall that in this problem there can be singularities of ψ_j at isolated points in the k-plane and that these somehow correspond to soliton components of the solution (according to Konopelchenko [6], for example). Now for the corresponding problem in $1+1$

dimensions, the focusing nonlinear Schrödinger equation, one has the result due to Klaus and Shaw [5] that if the initial phase function S vanishes identically and the amplitude function A is "bell-shaped" (only one critical point, a local maximum) and Lebsegue integrable, then the singularities are confined to the imaginary axis in the k-plane, and their number is asymptotically proportional to ϵ^{-1}. Moreover, they accumulate in a fixed interval of the imaginary axis with a density that is given explicitly in terms of A by a kind of Weyl formula [8]. It turns out that the solution of the focusing nonlinear Schrödinger equation is virtually "all solitons" in the sense that the reflection coefficient tends to zero also in the semiclassical limit [4]. This makes one wonder whether there might be a similar phenomenon for the focusing DS-II equation. To get started, one obviously would need numerical methods capable of detecting and locating singularities in the k-plane of the solutions ψ_j of the direct problem. This seems hard, but important in the longer term. I thought I'd mention it just to have it written down somewhere.

Here are some other numerical experiments that might be interesting.

(1) Of course one could directly try numerical simulation of the initial-value problem for the DS-II equation with initial data of the form (3) with several small values of ϵ. One could try these experiments for the same initial conditions and values of ϵ as proposed in the above list of problems regarding the direct scattering transform. The key would be to examine the solution over a region of the (x, y)-plane and over a time interval that is fixed while ϵ varies (decreases to zero). To get a feel for the solution one could plot either $|q|^2$ or $M = \Re\{g\}$ as a function of (x, y) for various times t independent of ϵ. What sort of features appear in the solution? Can the onset of some of these features be tied to the formation of singularities in the ϵ-independent solution of the dispersionless DS-II system (11)? Some work in this direction is already being carried out by Christian Klein, according to Ken.

(2) One should also try to study the equation (30) for the WKB exponent f subject to the asymptotic condition (33). This is a problem that depends on k but is independent of ϵ. Perhaps given $k \in \mathbb{C}$ this problem can be solved first in a neighborhood of $z = \infty$, and the solution can be numerically continued for decreasing $|z|$ until a singularity is encountered.

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