Introduction to Semiclassical Asymptotic Analysis: Lecture 2

Peter D. Miller

Department of Mathematics
University of Michigan

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Review

Solution of defocusing nonlinear Schrödinger by the inverse-scattering transform

We are interested in the initial-value problem

$$i\epsilon\psi_t^{\epsilon} + \frac{\epsilon^2}{2}\psi_{xx}^{\epsilon} - |\psi^{\epsilon}|^2\psi^{\epsilon} = 0, \;\; \psi^{\epsilon}(x,0) = \psi_0^{\epsilon}(x) := \sqrt{\rho_0(x)}e^{iS(x)/\epsilon},$$

where

$$S(x) := \int_0^x u_0(y)\,dy.$$

We assume that ρ_0 and u_0' are Schwartz-class functions.

$$\begin{tikzcd}
\boxed{\psi^{\epsilon}(x,0)} \arrow[r, "i\epsilon\psi_t^{\epsilon} + \frac{1}{2}\epsilon^2\psi_{xx}^{\epsilon} - |\psi^{\epsilon}|^2\psi^{\epsilon} = 0"] \arrow[d, "\mathscr{S}"'] & \boxed{\psi^{\epsilon}(x,t)} \\
\boxed{R_0^{\epsilon}(\lambda)} \arrow[r, "e^{2i\lambda^2 t/\epsilon}"'] & \boxed{e^{2i\lambda^2 t/\epsilon}R_0^{\epsilon}(\lambda)} \arrow[u, "\mathscr{S}^{-1}"']
\end{tikzcd}$$

Recall that the initial-value problem implies for the real fields

$$\rho^{\epsilon} := |\psi^{\epsilon}|^2 \text{ and } u^{\epsilon} := \Im\left\{\frac{\epsilon\psi_x^{\epsilon}}{\psi^{\epsilon}}\right\} \quad \text{the system}$$

$$\frac{\partial\rho^{\epsilon}}{\partial t} + \frac{\partial}{\partial x}(\rho^{\epsilon}u^{\epsilon}) = 0 \quad \text{and} \quad \frac{\partial u^{\epsilon}}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^{\epsilon 2} + \rho^{\epsilon}\right) = \frac{1}{2}\epsilon^2\frac{\partial F[\rho^{\epsilon}]}{\partial x}$$

with initial conditions $\rho^{\epsilon}(x,0) = \rho_0(x)$ and $u^{\epsilon}(x,0) = u_0(x)$, where

$$F[\rho] := \frac{1}{2\rho}\frac{\partial^2\rho}{\partial x^2} - \left(\frac{1}{2\rho}\frac{\partial\rho}{\partial x}\right)^2.$$

Neglecting $\epsilon^2 F_x$ leads to an ϵ-independent initial-value problem for the nonlinear hyperbolic *dispersionless defocusing NLS system*.

This cannot be a global model of the semiclassical dynamics, due to shock formation.

We need to calculate the *Jost solution* $\mathbf{w}$ of the linear equation

$$\epsilon \frac{d\mathbf{w}}{dx} = \begin{bmatrix} -i\lambda & \sqrt{\rho_0(x)}e^{iS(x)/\epsilon} \\ \sqrt{\rho_0(x)}e^{-iS(x)/\epsilon} & i\lambda \end{bmatrix} \mathbf{w},$$

that is, the solution for $\lambda \in \mathbb{R}$ that is determined (assuming sufficiently rapid decay of ρ_0 for large $|x|$) by the conditions

$$\mathbf{w}(x) = \begin{bmatrix} e^{-i\lambda x/\epsilon} \\ 0 \end{bmatrix} + R_0^\epsilon(\lambda) \begin{bmatrix} 0 \\ e^{i\lambda x/\epsilon} \end{bmatrix} + o(1), \quad x \to +\infty$$

and

$$\mathbf{w}(x) = T_0^\epsilon(\lambda) \begin{bmatrix} e^{-i\lambda x/\epsilon} \\ 0 \end{bmatrix} + o(1), \quad x \to -\infty,$$

for some coefficients $R_0^\epsilon(\lambda)$ (the *reflection coefficient*) and $T_0^\epsilon(\lambda)$ (the *transmission coefficient*).

Under suitable additional conditions on ρ_0 and u_0, the reflection coefficient $R_0^{\epsilon}(\lambda)$ can be approximated accurately in the limit $\epsilon \to 0$ by

$$\tilde{R}_0^{\epsilon}(\lambda) := \chi_{[\lambda_-,\lambda_+]}(\lambda)\sqrt{1 - e^{-2\tau(\lambda)/\epsilon}}e^{-2i\Phi(\lambda)/\epsilon}$$

where $\lambda_- := \inf_{x\in\mathbb{R}} \alpha(x)$ and $\lambda_+ := \sup_{x\in\mathbb{R}} \beta(x)$ with

$$\alpha(x) := -\tfrac{1}{2}u_0(x) - \sqrt{\rho_0(x)} \quad \text{and} \quad \beta(x) := -\tfrac{1}{2}u_0(x) + \sqrt{\rho_0(x)}$$

and where

$$\tau(\lambda) := \int_{x_-(\lambda)}^{x_+(\lambda)} \sqrt{\rho_0(y) - (\lambda + \tfrac{1}{2}u_0(y))^2}\, dy$$

$$\Phi(\lambda) := \tfrac{1}{2}S(x_+(\lambda)) + \lambda x_+(\lambda) - \int_{x_+(\lambda)}^{+\infty} \left[\sigma\sqrt{(\lambda + \tfrac{1}{2}u_0(y))^2 - \rho_0(y)} - (\lambda + \tfrac{1}{2}u_0(y))\right] dy$$

$$\sigma := \operatorname{sgn}(\lambda + \tfrac{1}{2}u_0(+\infty)).$$

Solve (for each fixed x and t) the following Riemann-Hilbert problem: seek $\mathbf{M} : \mathbb{C} \setminus \mathbb{R} \to \mathrm{SL}(2, \mathbb{C})$ such that:

- **Analyticity**: $\mathbf{M}$ is analytic in each half-plane, and takes boundary values $\mathbf{M}_\pm : \mathbb{R} \to \mathrm{SL}(2, \mathbb{C})$ on the real line from $\mathbb{C}_\pm$.
- **Jump Condition**: The boundary values are related by

$$\mathbf{M}_+(\lambda) = \mathbf{M}_-(\lambda) \begin{bmatrix} 1 - |R_0^\epsilon(\lambda)|^2 & -e^{-2i(\lambda x + \lambda^2 t)/\epsilon} R_0^\epsilon(\lambda)^* \\ e^{2i(\lambda x + \lambda^2 t)/\epsilon} R_0^\epsilon(\lambda) & 1 \end{bmatrix}, \quad \lambda \in \mathbb{R}.$$

- **Normalization**: As $\lambda \to \infty$, $\mathbf{M}(\lambda) \to \mathbb{I}$.

The solution of the initial-value problem is given by

$$\psi^\epsilon(x, t) = 2i \lim_{\lambda \to \infty} \lambda M_{12}(\lambda).$$

Modification of Initial Data

We are going to formulate the Riemann-Hilbert problem with $\tilde{R}_0^{\epsilon}(\lambda)$ in place of $R_0^{\epsilon}(\lambda)$. This requires some comment because it is not obvious that $\tilde{R}_0^{\epsilon}(\lambda)$ is even in the image of $\mathscr{S}$.

The key point is that if $R_0^{\epsilon}(\lambda)$ is any function for which the Riemann-Hilbert problem can be solved, then the extracted potential $\psi^{\epsilon} := 2i \lim_{\lambda\to\infty} \lambda M_{12}(\lambda)$ is a solution of the defocusing nonlinear Schrödinger equation. Indeed, the matrix $\mathbf{W}(\lambda) := \mathbf{M}(\lambda)e^{i(\lambda x+\lambda^2 t)\sigma_3/\epsilon}$ satisfies jump conditions that are independent of x and t, and therefore

$$\mathbf{U} := \epsilon\frac{\partial \mathbf{W}}{\partial x}\mathbf{W}^{-1} \quad \text{and} \quad \mathbf{V} := \epsilon\frac{\partial \mathbf{W}}{\partial t}\mathbf{W}^{-1}$$

are entire functions of λ. In fact, one can check that $\mathbf{U}$ ($\mathbf{V}$) is a linear (quadratic) function. These are precisely the coefficient matrices in the Lax pair (cf. Lecture 1). The existence of $\mathbf{W}$ as a simultaneous fundamental solution matrix guarantees the compatibility, which implies that ψ^{ϵ} solves the desired nonlinear PDE.

Modification of Initial Data

We will show directly that after replacing $R_0^\epsilon(\lambda)$ by its approximation $\tilde{R}_0^\epsilon(\lambda)$:

- The Riemann-Hilbert problem can indeed be solved as long as ϵ is sufficiently small.
- When $t = 0$, the extracted potential ψ^ϵ is close in the limit $\epsilon \to 0$ to the actual initial data.

Of course ψ^ϵ is also an exact solution of the defocusing nonlinear Schrödinger equation.

Let Σ be an oriented contour (perhaps with self-intersection points), and let $\mathbf{V} : \Sigma \to \mathrm{SL}(2, \mathbb{C})$ be a given jump matrix decaying to $\mathbb{I}$ as $\lambda \to \infty$ along any unbounded arcs of Σ. A general Riemann-Hilbert problem is the following: find $\mathbf{M} : \mathbb{C} \setminus \Sigma \to \mathrm{SL}(2, \mathbb{C})$ such that:

- **Analyticity**: $\mathbf{M}$ is analytic in its domain of definition, and takes boundary values $\mathbf{M}_\pm : \Sigma \to \mathrm{SL}(2, \mathbb{C})$ on Σ from the left $(+)$ and right $(-)$.
- **Jump Condition**: The boundary values are related by $\mathbf{M}_+(\lambda) = \mathbf{M}_-(\lambda)\mathbf{V}(\lambda)$ for $\lambda \in \Sigma$.
- **Normalization**: As $\lambda \to \infty$, $\mathbf{M}(\lambda) \to \mathbb{I}$.

This problem can be studied by converting it into a linear system of singular integral equations.

Subtract $\mathbf{M}_{-}(\lambda)$ from both sides of the jump condition:

$$\mathbf{M}_{+}(\lambda) - \mathbf{M}_{-}(\lambda) = \mathbf{M}_{-}(\lambda)(\mathbf{V}(\lambda) - \mathbb{I}), \quad \lambda \in \Sigma.$$

Taking into account the analyticity of $\mathbf{M}$ in $\mathbb{C} \setminus \mathbb{I}$ and the asymptotic value of $\mathbb{I}$ as $\lambda \to \infty$ it is necessary that $\mathbf{M}(\lambda)$ is given by the Cauchy integral (Plemelj formula):

$$\mathbf{M}(\lambda) = \mathbb{I} + \frac{1}{2\pi i} \int_{\Sigma} \frac{\mathbf{M}_{-}(\mu)(\mathbf{V}(\mu) - \mathbb{I})}{\mu - \lambda} \, d\mu, \quad \lambda \in \mathbb{C} \setminus \Sigma.$$

Letting λ tend to Σ from the right we obtain a closed equation for the boundary value $\mathbf{M}_{-}(\lambda)$, $\lambda \in \Sigma$:

$$\mathbf{X}(\lambda) - \frac{1}{2\pi i} \int_{\Sigma} \frac{\mathbf{X}(\mu)(\mathbf{V}(\mu) - \mathbb{I})}{\mu - \lambda_{-}} \, d\mu = \frac{1}{2\pi i} \int_{\Sigma} \frac{\mathbf{V}(\mu) - \mathbb{I}}{\mu - \lambda_{-}} \, d\mu, \quad \lambda \in \Sigma,$$

where $\mathbf{X}(\lambda) := \mathbf{M}_{-}(\lambda) - \mathbb{I}$.

If the jump matrix $\mathbf{V}$ depends on parameters (e.g., x, t, ϵ), one can consider the asymptotic behavior of the Riemann-Hilbert problem with respect to one or more parameters. While one could attempt to analyze the singular equation, this would generally be a difficult (perhaps impossible) task, and we will proceed differently.

The singular integral equation is perhaps the most useful in the *small norm setting*. This means that $\mathbf{V} - \mathbb{I}$ is small in both the $L^2(\Sigma)$ and $L^\infty(\Sigma)$ sense. This is a consequence of the fact that for a general class of contours Σ, the operator

$$\mathbf{F} \mapsto \frac{1}{2\pi i} \int_\Sigma \frac{\mathbf{F}(\mu)\, d\mu}{\mu - \lambda_-}$$

is bounded on $L^2(\Sigma)$, with a norm that only depends on geometrical details of Σ. See McIntosh, Coifman, Meyer for Lipschitz arcs, and Beals and Coifman for self-intersection points.

For problems of small norm type, the following hold true:

- The singular integral equation can be solved in $L^2(\Sigma)$ by iteration (contraction mapping, or Neumann series). This guarantees existence and uniqueness of the solution.
- It also allows the solution to be constructed (approximated with arbitrary accuracy and estimated). The $L^2(\Sigma)$ norm of $\mathbf{X}$ is proportional to that of $\mathbf{V} - \mathbb{I}$.
- Under suitable other technical assumptions, $\mathbf{M}(\lambda)$ has an asymptotic expansion as $\lambda \to \infty$:

$$\mathbf{M}(\lambda) = \mathbb{I} + \sum_{n=1}^{N} \lambda^{-n}\mathbf{M}_n + O(\lambda^{-(N+1)}), \quad \lambda \to \infty$$

and the moments $\mathbf{M}_n$ are bounded in terms of norms of $\mathbf{V} - \mathbb{I}$.

The Riemann-Hilbert Problem for Semiclassical Defocusing NLS

Seek $\mathbf{M} : \mathbb{C} \setminus [\lambda_-, \lambda_+] \to \mathrm{SL}(2, \mathbb{C})$ with the following properties:

- **Analyticity**: $\mathbf{M}$ is analytic in its domain of definition and takes boundary values $\mathbf{M}_\pm(\lambda)$ on (λ_-, λ_+) from $\mathbb{C}_\pm$.
- **Jump Condition**: $\mathbf{M}_+(\lambda) = \mathbf{M}_-(\lambda)\mathbf{V}(\lambda)$ for $\lambda_- < \lambda < \lambda_+$, where

$$\mathbf{V}(\lambda) = \begin{bmatrix} e^{-2\tau(\lambda)/\epsilon} & -e^{-2i\theta(\lambda;x,t)/\epsilon}H^\epsilon(\lambda) \\ e^{2i\theta(\lambda;x,t)/\epsilon}H^\epsilon(\lambda) & 1 \end{bmatrix},$$

  $\theta(\lambda; x, t) := \lambda x + \lambda^2 t - \Phi(\lambda)$, and $H^\epsilon(\lambda) := \sqrt{1 - e^{-2\tau(\lambda)/\epsilon}}$.
- **Normalization**: As $\lambda \to \infty$, $\mathbf{M}(\lambda) \to \mathbb{I}$.

This is *not* a small-norm problem in the semiclassical limit $\epsilon \to 0$.

Outline of Method

We will apply the *steepest descent method* developed for Riemann-Hilbert problems by Deift and Zhou (for the first application to semiclassical asymptotics see Deift, Venakides, and Zhou, 1997). The general steps in the method are:

1. **Control of oscillations.** An analogue of the WKB exponent f is introduced, a scalar-valued analytic function $g : \mathbb{C} \setminus [\lambda_-, \lambda_+] \to \mathbb{C}$ with $g(\lambda) \to 0$ as $\lambda \to \infty$. The substitution $\mathbf{M}(\lambda) := \mathbf{N}(\lambda)e^{ig(\lambda)\sigma_3/\epsilon}$ implies an equivalent Riemann-Hilbert problem for $\mathbf{N}(\lambda)$ whose jump matrix involves the boundary values taken on $[\lambda_-, \lambda_+]$ by g. The function g is selected to control the oscillations in the off-diagonal elements of the jump matrix.
2. **Steepest descent.** Based on the control afforded by introduction of g, monotone phases are deformed into the complex plane with the help of two types of matrix factorizations. This step amounts to a substitution $\mathbf{N}(\lambda) := \mathbf{O}(\lambda)\mathbf{L}(\lambda)$, where $\mathbf{L}(\lambda)$ is an explicit, piecewise-analytic matrix function.

Outline of Method

3. **Parametrix construction.** The jump matrix for $\mathbf{O}$ has obvious asymptotics as $\epsilon \to 0$, suggesting a certain explicit approximation for $\mathbf{O}$, denoted $\dot{\mathbf{O}}(\lambda)$ and called a *parametrix*. The parametrix is defined as an *outer parametrix* away from exceptional points in the interval $[\lambda_-, \lambda_+]$. Near the exceptional points one installs certain *inner parametrices*.
4. **Error analysis by small norm theory.** We compare the unknown matrix $\mathbf{O}(\lambda)$ with its explicit parametrix $\dot{\mathbf{O}}(\lambda)$ by considering the *error matrix* $\mathbf{E}(\lambda) := \mathbf{O}(\lambda)\dot{\mathbf{O}}(\lambda)^{-1}$. While unknown, one uses properties of the explicit parametrix to show that $\mathbf{E}(\lambda)$ satisfies the conditions of a Riemann-Hilbert problem of small-norm type in the semiclassical limit $\epsilon \to 0$. This implies that $\mathbf{E} - \mathbb{I}$ is small as $\epsilon \to 0$.
5. **Extraction of the solution.** By unraveling the steps:

$$\mathbf{M}(\lambda) = \mathbf{N}(\lambda)e^{ig(\lambda)\sigma_3/\epsilon} = \mathbf{O}(\lambda)\mathbf{L}(\lambda)e^{ig(\lambda)\sigma_3/\epsilon} = \mathbf{E}(\lambda)\dot{\mathbf{O}}(\lambda)\mathbf{L}(\lambda)e^{ig(\lambda)\sigma_3/\epsilon}$$

where only the error term is not explicit. Then extract $\psi^\epsilon(x,t)$ by

$$\psi^\epsilon(x,t) = 2i \lim_{\lambda\to\infty} \lambda M_{12}(\lambda).$$

We're on a boat. . . I'm your captain. . . join me now.
—Tom Waits, Big Time

Step 1: Control of Oscillations

Choice of g-function

Making the substitution $\mathbf{M}(\lambda) = \mathbf{N}(\lambda)e^{ig(\lambda)\sigma_3/\epsilon}$, and using the facts that $g : \mathbb{C} \setminus [\lambda_-, \lambda_+] \to \mathbb{C}$ is analytic and $g(\infty) = 0$, we can easily see that $\mathbf{N} : \mathbb{C} \setminus [\lambda_-, \lambda_+] \to \mathrm{SL}(2, \mathbb{C})$ satisfies the conditions of this Riemann-Hilbert problem:

- **Analyticity**: $\mathbf{N}$ is analytic in $\mathbb{C} \setminus [\lambda_-, \lambda_+]$, taking boundary values $\mathbf{N}_\pm(\lambda)$ on $[\lambda_-, \lambda_+]$ from $\mathbb{C}_\pm$.
- **Jump Condition**: The boundary values are related by $\mathbf{N}_+(\lambda) = \mathbf{N}_-(\lambda)\mathbf{V}^{(\mathbf{N})}(\lambda)$ for $\lambda_- < \lambda < \lambda_+$, where

$$\mathbf{V}^{(\mathbf{N})}(\lambda) := \begin{bmatrix} e^{2(\Delta(\lambda)-\tau(\lambda))/\epsilon} & -e^{-2i\phi(\lambda)/\epsilon}H^\epsilon(\lambda) \\ e^{2i\phi(\lambda)/\epsilon}H^\epsilon(\lambda) & e^{-2\Delta(\lambda)/\epsilon} \end{bmatrix},$$

  $2\Delta(\lambda) := -i(g_+(\lambda) - g_-(\lambda))$, and $2\phi(\lambda) := 2\theta(\lambda) - g_+(\lambda) - g_-(\lambda)$.
- **Normalization**: As $\lambda \to \infty$, $\mathbf{N}(\lambda) \to \mathbb{I}$.

We suppose that $g(\lambda) = g(\lambda^*)^*$ can be found so that (λ_-, λ_+) can be partitioned into three types of subintervals:

- **Voids**: These are characterized by the conditions $\Delta(\lambda) \equiv 0$ and $\phi'(\lambda) > 0$.
- **Bands**: These are characterized by the conditions $0 < \Delta(\lambda) < \tau(\lambda)$ and $\phi'(\lambda) \equiv 0$.
- **Saturated regions**: These are characterized by the conditions $\Delta(\lambda) \equiv \tau(\lambda)$ and $\phi'(\lambda) < 0$.

We sometimes collectively refer to voids and saturated regions as *gaps*. We always assume that gaps are separated by bands, and that the left and right-most subintervals of (λ_-, λ_+) are gaps.

We now examine the consequences of each type of interval for the jump matrix $\mathbf{V}^{(\mathbf{N})}(\lambda)$.

Under the condition that $\Delta(\lambda) \equiv 0$, the jump matrix $\mathbf{V}^{(\mathbf{N})}(\lambda)$ has an "upper-lower" factorization:

$$\mathbf{V}^{(\mathbf{N})}(\lambda) = \begin{bmatrix} 1 & -e^{-2i\phi(\lambda)/\epsilon}H^{\epsilon}(\lambda) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ e^{2i\phi(\lambda)/\epsilon}H^{\epsilon}(\lambda) & 1 \end{bmatrix}.$$

Let us assume that $\phi(\lambda)$ and $\tau(\lambda)$ are analytic (this will be the case if the initial data functions u_0 and ρ_0 are analytic). Then the condition $\phi'(\lambda) > 0$ makes $\phi(\lambda)$ a real analytic function that is strictly increasing in the void interval. By the Cauchy-Riemann equations, it follows that the imaginary part of $\phi(\lambda)$ is positive (negative) in the upper (lower) half-plane.

This implies that the first (second) matrix factor has an analytic continuation into the lower (upper) half-plane that is exponentially close to the identity matrix in the limit $\epsilon \to 0$.

The strict inequalities $0 < \Delta(\lambda) < \tau(\lambda)$ imply that the diagonal elements of $\mathbf{V}^{(\mathbf{N})}(\lambda)$, namely

$$e^{2(\Delta(\lambda)-\tau(\lambda))/\epsilon} \quad \text{and} \quad e^{-2\Delta(\lambda)/\epsilon}$$

are both exponentially small in the semiclassical limit $\epsilon \to 0$. The condition $\phi'(\lambda) \equiv 0$ together with the inequality $\tau(\lambda) > 0$ that holds for all $\lambda \in (\lambda_-, \lambda_+)$ then implies that $\mathbf{V}^{(\mathbf{N})}(\lambda)$ is exponentially close in the semiclassical limit to a constant off-diagonal matrix:

$$\mathbf{V}^{(\mathbf{N})}(\lambda) = \begin{bmatrix} 0 & -e^{-2i\phi/\epsilon} \\ e^{2i\phi/\epsilon} & 0 \end{bmatrix} + \text{exponentially small terms.}$$

The real constant ϕ can be different for different bands, and it generally can depend on x and t (but not ϵ).

Under the condition that $\Delta(\lambda) \equiv \tau(\lambda)$, the jump matrix $\mathbf{V}^{(\mathbf{N})}(\lambda)$ has a "lower-upper" factorization:

$$\mathbf{V}^{(\mathbf{N})}(\lambda) = \begin{bmatrix} 1 & 0 \\ e^{2i\phi(\lambda)/\epsilon}H^{\epsilon}(\lambda) & 1 \end{bmatrix} \begin{bmatrix} 1 & -e^{-2i\phi(\lambda)/\epsilon}H^{\epsilon}(\lambda) \\ 0 & 1 \end{bmatrix}.$$

The condition $\phi'(\lambda) < 0$ makes $\phi(\lambda)$ a real analytic function that is strictly decreasing in the void interval. By the Cauchy-Riemann equations, it follows that the imaginary part of $\phi(\lambda)$ is negative (positive) in the upper (lower) half-plane.

This again implies that the first (second) matrix factor has an analytic continuation into the lower (upper) half-plane that is exponentially close to the identity matrix in the limit $\epsilon \to 0$.

Let us construct g by temporarily setting aside the inequalities. Suppose that there are $N+1$ bands in (λ_-, λ_+) that we will denote by (a_j, b_j) with $a_0 < b_0 < a_1 < b_1 < \cdots < a_N < b_N$. The complementary intervals are either voids or saturated regions.

Recall that the boundary values of g are subject to the following conditions:

- $g_+(\lambda) - g_-(\lambda) = 0$ which implies $g'_+(\lambda) - g'_-(\lambda) = 0$ for λ in voids and outside of $[\lambda_-, \lambda_+]$.
- $g'_+(\lambda) + g'_-(\lambda) = 2\theta'(\lambda)$ for λ in bands.
- $g_+(\lambda) - g_-(\lambda) = 2i\tau(\lambda)$ which implies $g'_+(\lambda) - g'_-(\lambda) = 2i\tau'(\lambda)$ for λ in saturated regions.

We therefore know $g'_+ - g'_-$ everywhere along $\mathbb{R}$ with the exception of the band intervals, where we know instead $g'_+ + g'_-$.

Consider the function $r(\lambda)$ defined as follows:

- $r(\lambda)^2 = \prod_{n=0}^{N} (\lambda - a_n)(\lambda - b_n)$
- $r(\lambda)$ is analytic for $\lambda \in \mathbb{C} \setminus \bigcup_{n=0}^{N} [a_n, b_n]$.
- $r(\lambda) = \lambda^{N+1} + O(\lambda^N)$ as $\lambda \to \infty$.

The boundary values of r on any band satisfy $r_+(\lambda) + r_-(\lambda) = 0$. Consider instead of $g'(\lambda)$ the function $k(\lambda) := g'(\lambda)/r(\lambda)$. This function is analytic where g' is and satisfies

$$k_+(\lambda) - k_-(\lambda) = \begin{cases} 0, & \lambda \text{ in voids or outside of } [\lambda_-, \lambda_+] \\ \dfrac{2\theta'(\lambda)}{r_+(\lambda)}, & \lambda \text{ in bands} \\ \dfrac{2i\tau'(\lambda)}{r(\lambda)}, & \lambda \text{ in saturated regions.} \end{cases}$$

Step 1: Control of Oscillations

Formula for g

Up to an entire function (which must be zero for consistency with $g'(\lambda) = O(\lambda^{-2})$ as $\lambda \to \infty$), k must be given by the Plemelj formula:

$$k(\lambda) = \frac{1}{\pi i}\int_{\text{bands}} \frac{\theta'(\mu)\,d\mu}{r_+(\mu)(\mu-\lambda)} + \frac{1}{\pi}\int_{\text{saturated regions}} \frac{\tau'(\mu)\,d\mu}{r(\mu)(\mu-\lambda)}, \quad \text{so,}$$

$$g'(\lambda) = \frac{r(\lambda)}{\pi i}\int_{\text{bands}} \frac{\theta'(\mu)\,d\mu}{r_+(\mu)(\mu-\lambda)} + \frac{r(\lambda)}{\pi}\int_{\text{saturated regions}} \frac{\tau'(\mu)\,d\mu}{r(\mu)(\mu-\lambda)}.$$

The additional condition $g'(\lambda) = O(\lambda^{-2})$ as $\lambda \to \infty$ is equivalent to $k(\lambda) = O(\lambda^{-(N+3)})$. But since $(\mu-\lambda)^{-1} \sim -\lambda^{-1} - \mu\lambda^{-2} - \mu^2\lambda^{-3} + \cdots$,

$k(\lambda)$ has the Laurent series $\quad k(\lambda) = k_1\lambda^{-1} + k_2\lambda^{-2} + k_3\lambda^{-3} + \cdots \quad$ where

$$k_n := -\frac{1}{\pi i}\int_{\text{bands}} \frac{\theta'(\mu)\mu^{n-1}\,d\mu}{r_+(\mu)} - \frac{1}{\pi}\int_{\text{saturated regions}} \frac{\tau'(\mu)\mu^{n-1}\,d\mu}{r(\mu)}.$$

We therefore require that $k_n = 0$ for $n = 1, \ldots, N+2$.

With the conditions $k_1 = \cdots = k_{N+2} = 0$, we can obtain $g(\lambda)$ from $g'(\lambda)$ by contour integration:

$$g(\lambda) = \int_{\infty}^{\lambda} g'(\mu)\, d\mu \quad \text{because } g' \text{ is integrable at } \infty.$$

Note, however, that while we have arranged that $g'_+ - g'_- = 0$ in voids and $g'_+ - g'_- = 2i\tau'$ in saturated regions, we need to get integration constants correct to guarantee $g_+ - g_- = 0$ in voids and $g_+ - g_- = 2i\tau$ in saturated regions.

Since $\tau(\lambda_\pm) = 0$, one can check that the integration constants are automatically correct in the exterior gaps (λ_-, a_0) and (b_N, λ_+). There remains one condition to impose for each of the N interior gaps.

Step 1: Control of Oscillations

Formula for g

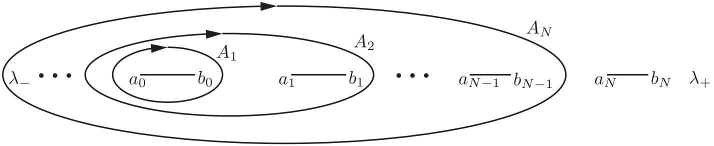
One can check that:

- If (b_n, a_{n+1}) is a void, then $g_+ - g_- = 0$ in this interval is equivalent to the contour integral condition

$$\oint_{A_{n+1}} g'(\lambda)\, d\lambda = 0.$$

- If (b_n, a_{n+1}) is a saturated region, then $g_+ - g_- = 2i\tau$ in this interval is equivalent to the contour integral condition

$$\oint_{A_{n+1}} [T'(\lambda) - g'(\lambda)]\, d\lambda = 0, \quad \text{where} \quad T(\lambda) := \frac{1}{\pi}\int_{\lambda_-}^{\lambda_+} \frac{\tau(\mu)\, d\mu}{\mu - \lambda}.$$



Step 1: Control of Oscillations

Formula for g

In total, we have assembled $2N + 2$ conditions on $2N + 2$ unknowns $a_0, b_0, \ldots, a_N, b_N$. If these equations have a unique solution, then associated with the symbol sequence $(s_0, s_1, \ldots, s_{N+1})$, $s_n =$ V or $s_n =$ S, indicating the types of the gaps in left-to-right order, we have determined $g(\lambda)$.

Of course this analysis has ignored the inequalities that the boundary values of g are supposed to satisfy. These inequalities should select:

- The genus N.
- The symbol sequence $(s_0, \ldots, s_{N+1})$.

The procedure in practice is therefore to determine N and $(s_0, \ldots, s_{N+1})$ so that the inequalities are true. The independent variables x and t are parameters in this procedure. In particular, the genus N will depend on (x, t).

Step 2: Steepest Descent

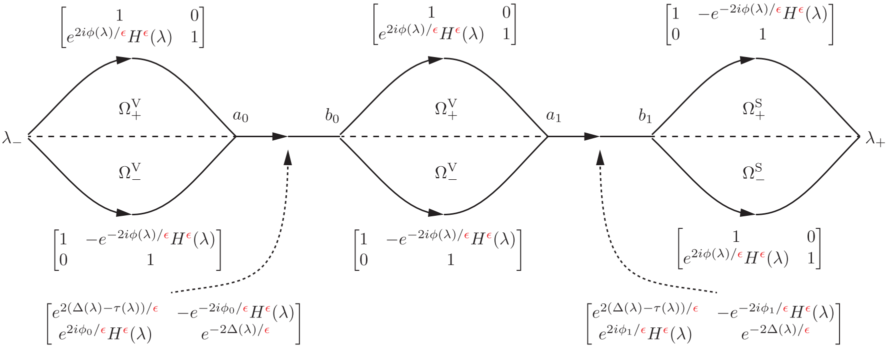
Opening lenses

Let us suppose that we have found a g-function. We now make a substitution to exploit the matrix factorizations designed for use in the gaps. Let $\Omega_\pm^{\mathrm{V}}$ ($\Omega_\pm^{\mathrm{S}}$) denote the union of thin lens-shaped domains in $\mathbb{C}_\pm$ that abut voids (saturated regions). Define the piecewise analytic matrix function $\mathbf{L}$ by

$$\mathbf{L}(\lambda) := \begin{cases} \begin{bmatrix} 1 & 0 \\ e^{2i\phi(\lambda)/\epsilon}H^\epsilon(\lambda) & 1 \end{bmatrix}, & \lambda \in \Omega_+^{\mathrm{V}}, \\ \begin{bmatrix} 1 & 0 \\ -e^{2i\phi(\lambda)/\epsilon}H^\epsilon(\lambda) & 1 \end{bmatrix}, & \lambda \in \Omega_-^{\mathrm{S}}, \\ \begin{bmatrix} 1 & e^{-2i\phi(\lambda)/\epsilon}H^\epsilon(\lambda) \\ 0 & 1 \end{bmatrix}, & \lambda \in \Omega_-^{\mathrm{V}} \\ \begin{bmatrix} 1 & -e^{-2i\phi(\lambda)/\epsilon}H^\epsilon(\lambda) \\ 0 & 1 \end{bmatrix}, & \lambda \in \Omega_+^{\mathrm{S}} \\ \mathbb{I}, & \text{otherwise.} \end{cases}$$

Make the substitution $\mathbf{N}(\lambda) = \mathbf{O}(\lambda)\mathbf{L}(\lambda)$. Then $\mathbf{O}(\lambda)$ satisfies:

- **Analyticity**: $\mathbf{O}$ is analytic in $\mathbb{C} \setminus \Sigma^{(\mathbf{O})}$, taking boundary values $\mathbf{O}_+$ ($\mathbf{O}_-$) on each oriented arc of $\Sigma^{(\mathbf{O})}$ from the left (right).
- **Jump Condition**: The boundary values are related by $\mathbf{O}_+(\lambda) = \mathbf{O}_-(\lambda)\mathbf{V}^{(\mathbf{O})}$ for $\lambda \in \Sigma^{(\mathbf{O})}$ (see figure).
- **Normalization**: As $\lambda \to \infty$, $\mathbf{O}(\lambda) \to \mathbb{I}$.



Step 3: Parametrix Construction

Outer parametrix

Letting $\epsilon \to 0$ pointwise in λ along $\Sigma^{(\mathbf{O})}$, the jump matrix $\mathbf{V}^{(\mathbf{O})}(\lambda)$ converges to $\mathbb{I}$, except along the band (a_n, b_n), where

$$\mathbf{V}^{(\mathbf{O})}(\lambda) = \begin{bmatrix} 0 & -e^{-2i\phi_n/\epsilon} \\ e^{2i\phi_n/\epsilon} & 0 \end{bmatrix} + \text{exponentially small terms}$$

where ϕ_n are well-defined real-valued functions of (x, t) that are independent of λ and ϵ. This suggests a formal approximation for $\mathbf{O}(\lambda)$ that solves the following problem: seek $\dot{\mathbf{O}}^{(\mathrm{out})} : \mathbb{C} \setminus \text{bands} \to \mathrm{SL}(2, \mathbb{C})$ with the properties

- **Analyticity**: $\dot{\mathbf{O}}^{(\mathrm{out})}$ is analytic where defined and takes boundary values $\dot{\mathbf{O}}_{\pm}^{(\mathrm{out})}(\lambda)$ from $\mathbb{C}_{\pm}$ on each band (a_n, b_n).
- **Jump Condition**: The boundary values satisfy $(n = 0, \ldots, N)$

$$\dot{\mathbf{O}}_{+}^{(\mathrm{out})}(\lambda) = \dot{\mathbf{O}}_{-}^{(\mathrm{out})}(\lambda) \begin{bmatrix} 0 & -e^{-2i\phi_n/\epsilon} \\ e^{2i\phi_n/\epsilon} & 0 \end{bmatrix}, \quad a_n < \lambda < b_n.$$

- **Normalization**: As $\lambda \to \infty$, $\dot{\mathbf{O}}^{(\mathrm{out})}(\lambda) \to \mathbb{I}$.

Step 3: Parametrix Construction

Outer parametrix

Since the jump matrix is discontinuous at the band endpoints, we need to specify a singularity at each; we will suppose that for all n,

$$\dot{\mathbf{O}}^{(\mathrm{out})}(\lambda) = O((\lambda - a_n)^{-1/4}(\lambda - b_n)^{-1/4}), \quad \lambda \to a_n, b_n.$$

With this condition, there is a unique solution for $\dot{\mathbf{O}}^{(\mathrm{out})}(\lambda)$ that we call the *outer parametrix*. In general, it is constructed in terms of Riemann theta functions of genus N, but for $N = 0$ (one band) the solution is elementary:

$$\dot{\mathbf{O}}^{(\mathrm{out})}(\lambda) = e^{-i\phi_0\sigma_3/\epsilon}\mathbf{A}\gamma(\lambda)^{\sigma_3}\mathbf{A}^{-1}e^{i\phi_0\sigma_3/\epsilon}, \quad \text{where} \quad \mathbf{A} := \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$$

and where $\gamma(\lambda)$ is the function analytic for $\lambda \in \mathbb{C} \setminus [a_0, b_0]$ that satisfies

$$\gamma(\lambda)^4 = \frac{\lambda - b_0}{\lambda - a_0} \quad \text{and} \quad \lim_{\lambda \to \infty} \gamma(\lambda) = 1.$$

Step 3: Parametrix Construction

Inner parametrices

The approximation of the jump matrix $\mathbf{V}^{(\mathbf{O})}(\lambda)$ leading to the outer parametrix fails to be uniformly valid near the band endpoints.

Elsewhere the accuracy is uniformly of order $O(1/\log(\epsilon^{-1}))$, dominated by behavior near $\lambda_{\pm}$. Away from these points we have exponential accuracy.

Therefore, it it reasonable that another approximation of $\mathbf{O}(\lambda)$ will need to be constructed for each small disk centered at a band endpoint. The recipe for this construction is the following:

1. Replace the three jump matrices locally by exponentially accurate approximations by replacing $H^{\epsilon}(\lambda)$ with 1 and dropping the uniformly exponentially small diagonal entry of $\mathbf{V}^{(\mathbf{O})}(\lambda)$ on the band.

Step 3: Parametrix Construction

Inner parametrices

2. Find a matrix that locally solves the resulting jump conditions exactly.
    - Use conformal mapping $\lambda \to \zeta$ to simplify the exponents (goal: make them all proportional to $\zeta^{3/2}$).
    - Solve the simplified jump conditions with the help of Airy functions.
3. Multiply the solution on the left by a matrix holomorphic near the band endpoint (which cannot alter the jump conditions) chosen to match well onto the outer parametrix on the disk boundary.

The result of this procedure is a matrix function $\dot{\mathbf{O}}^{(\mathrm{in},D)}(\lambda)$ called an *inner parametrix* defined in an ϵ-independent disk D containing the band endpoint of interest with the following properties:

- $\dot{\mathbf{O}}^{(\mathrm{in},D)}(\lambda) = O(\epsilon^{-1/6})$ uniformly for $\lambda \in D$.
- $\dot{\mathbf{O}}_{+}^{(\mathrm{in},D)}(\lambda) = \dot{\mathbf{O}}_{-}^{(\mathrm{in},D)}(\lambda)(\mathbb{I} + \text{exponentially small})\mathbf{V}^{(\mathbf{O})}(\lambda)$ for $\lambda \in \Sigma^{(\mathbf{O})} \cap D$.
- $\dot{\mathbf{O}}^{(\mathrm{in},D)}(\lambda)\dot{\mathbf{O}}^{(\mathrm{out})}(\lambda)^{-1} = \mathbb{I} + O(\epsilon)$ uniformly for $\lambda \in \partial D$.

Step 3: Parametrix Construction

Global parametrix

Each band endpoint gets its own disk $D_{a_0}, \ldots, D_{b_N}$, and its own inner parametrix. Combining these definitions with the outer parametrix gives rise to an explicit, ad-hoc approximation of $\mathbf{O}(\lambda)$ called the *global parametrix* denoted $\dot{\mathbf{O}}(\lambda)$ and defined as follows:

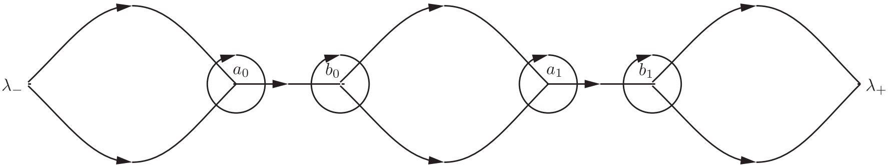
$$\dot{\mathbf{O}}(\lambda) := \begin{cases} \dot{\mathbf{O}}^{(\mathrm{in},D_p)}(\lambda), & \lambda \in D_p, \quad p = a_0, \ldots, b_N, \\ \dot{\mathbf{O}}^{(\mathrm{out})}(\lambda), & \text{otherwise.} \end{cases}$$

Step 4: Error Analysis by Small Norm Theory

Let the *error* of the approximation be defined as the matrix function

$$\mathbf{E}(\lambda) := \mathbf{O}(\lambda)\dot{\mathbf{O}}(\lambda)^{-1}$$

wherever both factors make sense. This makes $\mathbf{E}(\lambda)$ analytic on the complement of an arcwise oriented contour $\Sigma^{(\mathbf{E})}$ (pictured).



While $\mathbf{O}$ is only specified as the solution of a Riemann-Hilbert problem, the global parametrix $\dot{\mathbf{O}}(\lambda)$ is known. Therefore we may regard the mapping $\mathbf{O} \to \mathbf{E}$ as a substitution resulting in an equivalent Riemann-Hilbert problem for $\mathbf{E}$.

Step 4: Error Analysis by Small Norm Theory

Since both $\mathbf{O}(\lambda) \to \mathbb{I}$ (by normalization condition) and $\dot{\mathbf{O}}(\lambda) \to \mathbb{I}$ (by construction) as $\lambda \to \infty$, we also must have $\mathbf{E}(\lambda) \to \mathbb{I}$ in this limit.

By direct calculations, one checks that as a consequence of the uniform boundedness of the outer parametrix outside all disks,

$$\mathbf{E}_+(\lambda) = \mathbf{E}_-(\lambda)(\mathbb{I} + O(1/\log(\epsilon^{-1}))) \quad \text{uniformly for } \lambda \in \Sigma^{(\mathbf{E})}.$$

This means that $\mathbf{E}(\lambda)$ satisfies the conditions of a Riemann-Hilbert problem of small norm type, with estimates of $\mathbf{V}^{(\mathbf{E})}(\lambda) - \mathbb{I}$ in all required spaces being $O(1/\log(\epsilon^{-1}))$. Small-norm theory therefore implies that:

- $\mathbf{E}(\lambda)$ exists for sufficiently small ϵ and is unique, and hence (by unraveling the explicit substitutions) the same is true of $\mathbf{M}(\lambda)$.
- $\mathbf{E}(\lambda)$ has a Laurent series (convergent, because $\Sigma^{(\mathbf{E})}$ is bounded)

$$\mathbf{E}(\lambda) = \mathbb{I} + \sum_{n=1}^{\infty} \mathbf{E}_n \lambda^{-n} \quad \text{with} \quad \mathbf{E}_n = O(1/\log(\epsilon^{-1})), \quad \forall n.$$

Step 5: Extraction of the Solution

Recall that a solution of the defocusing nonlinear Schrödinger equation $\psi^\epsilon(x,t)$ is obtained from the (well-defined for sufficiently small ϵ) solution $\mathbf{M}(\lambda)$ of the original Riemann-Hilbert problem of inverse scattering with modified reflection coefficient via

$$\psi^\epsilon(x,t) = 2i \lim_{\lambda\to\infty} \lambda M_{12}(\lambda).$$

Now we express this in terms of known quantities and the error matrix $\mathbf{E}$. Since $\mathbf{L}(\lambda) = \mathbb{I}$ and $\dot{\mathbf{O}}(\lambda) = \dot{\mathbf{O}}^{(\mathrm{out})}(\lambda)$ both hold for large enough $|\lambda|$,

$$\begin{aligned} \psi^\epsilon(x,t) &= 2i \lim_{\lambda\to\infty} \left[\mathbf{E}(\lambda)\dot{\mathbf{O}}^{(\mathrm{out})}(\lambda)e^{ig(\lambda)\sigma_3/\epsilon}\right]_{12} \\ &= 2iE_{1,12} + 2i\dot{O}^{(\mathrm{out})}_{1,12} \\ &= 2i\dot{O}^{(\mathrm{out})}_{1,12} + O(1/\log(\epsilon^{-1})). \end{aligned}$$

When $N=0$ (one band, (a_0,b_0)), this reads simply

$$\psi^\epsilon(x,t) = \frac{1}{2}(b_0-a_0)e^{-2i\phi_0/\epsilon} + O(1/\log(\epsilon^{-1})), \qquad \frac{\partial\phi_0}{\partial x} = \frac{1}{2}(a_0+b_0).$$

The g-Function When $t = 0$

We want to give some further details of this procedure in some simple cases. We first claim that the g-function can be determined explicitly when $t = 0$, and that $N = 0$ (one band) suffices in this case.

Recall that for $N = 0$ there are just two conditions to be satisfied by the endpoints a_0, b_0: $k_1 = k_2 = 0$. We have the following result

Proposition

Set $t = 0$. The equations $k_1 = k_2 = 0$ are simultaneously satisfied by

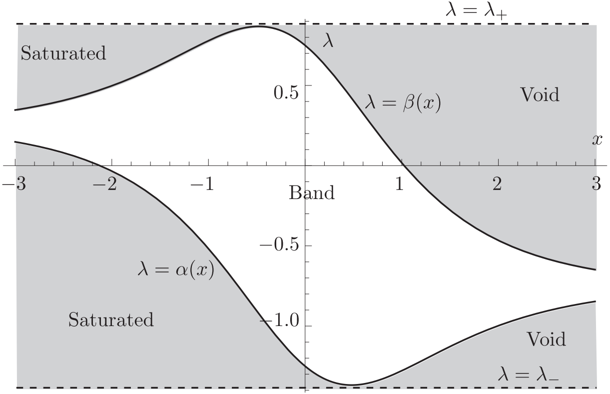
$$a_0 = \alpha(x) \quad \textit{and} \quad b_0 = \beta(x)$$

with symbol sequences

- (V, V) *where* $\alpha'(x) > 0$ *and* $\beta'(x) < 0$
- (V, S) *where* $\alpha'(x) > 0$ *and* $\beta'(x) > 0$
- (S, V) *where* $\alpha'(x) < 0$ *and* $\beta'(x) < 0$
- (S, S) *where* $\alpha'(x) < 0$ *and* $\beta'(x) > 0$.

The g-Function When $t = 0$

One can further confirm that the necessary inequalities are also satisfied by the specified configuration when $t = 0$. This information is summarized in this figure:



Perturbation Theory for Small Time

The implicit function theorem can be used to continue the solution to $k_1 = k_2 = 0$ for small t independent of ϵ. The necessary inequalities also persist as they hold strictly when $t = 0$. Therefore we have a genus $N = 0$ configuration of a single band for all $x \in \mathbb{R}$ if t is sufficiently small.

Implicit differentiation of the conditions $k_1 = 0$ and $k_2 = 0$ with respect to x and t shows that the following equations hold true:

$$\frac{\partial a_0}{\partial t} - \left[\frac{3}{2}a_0 + \frac{1}{2}b_0\right]\frac{\partial a_0}{\partial x} = 0 \quad \text{and} \quad \frac{\partial b_0}{\partial t} - \left[\frac{3}{2}b_0 + \frac{1}{2}a_0\right]\frac{\partial b_0}{\partial x} = 0.$$

Note that setting $a_0 = -\frac{1}{2}u - \sqrt{\rho}$ and $b_0 = -\frac{1}{2}u + \sqrt{\rho}$ this system becomes the dispersionless defocusing NLS system

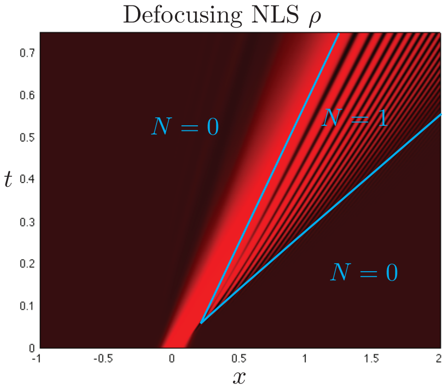
$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 \quad \text{and} \quad \frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2 + \rho\right) = 0.$$

Also, $\psi^{\epsilon}(x,t) = \sqrt{\rho(x,t)}e^{i\int^x u(y,t)\,dy/\epsilon} + O(1/\log(\epsilon^{-1}))$.

Bifurcation Theory

Jumping genus, Batman!

For larger t, the g-function theory tiles the (x,t)-plane with regions corresponding to different genera N. The earliest point of transition is the shock time for the dispersionless NLS system.



$$\rho_0(x) = \frac{1}{10} + \frac{1}{2}e^{-256x^2}$$
$$u_0(x) = 1$$
$$\epsilon = 0.0122$$

Periodic boundary conditions

Genus bifurcations in the g-function are the integrable nonlinear analogues of stationary phase point bifurcations in the linear theory.