IST versus PDE's

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Aim of the talk

- Comparison of (rigorous) results obtained by Inverse Scattering Transform (IST) methods and PDE methods, mostly in 2D.
- What can we learn (on more general equations) from integrable equations.
- Comments on various aspects of the rigorous justification of asymptotic equations or systems.

Preliminary remarks

- The classical dispersive equations or systems (Korteweg-de Vries, Benjamin-Ono, Kadomtsev-Petviashvili, Boussinesq, Davey-Stewartson, nonlinear Schrödinger, Green-Naghdi,..) are not derived from first principles but as asymptotics models derived from more complex systems, on a suitable regime of amplitude, wavelengths, wave steepness...

  Those asymptotic models are not supposed to be good approximations for all time scales, and for instance, the classical dichotomy "blow-up in finite time /global well-posedness" is not always very relevant here and should be replaced by "long time" issues (with the corresponding uniform bounds and error estimates with the solutions of the original system).
- It turns out that a few of them are integrable by Inverse Scattering techniques, which can provide useful insights on the dynamics on the non integrable ones.
- As a paradigm, we will consider the water wave system from which most of the classical nonlinear dispersive equations and systems can be derived (as asymptotic models), and give a few details on the derivation of the Davey-Stewartson systems.

- The water waves system (derived by Lagrange in 1781) is a typical example of a very relevant, but mathematically complex, physical system which leads in various asymptotics limits to most of the well-known nonlinear dispersive equations or systems.
- One needs first to define small parameters :
  a = typical amplitude, h = typical depth, λ = typical horizontal wavelength (in the isotropic case, if not λ_x, λ_y).

$$\epsilon = \frac{a}{h}, \quad \mu = \left(\frac{h}{\lambda}\right)^2, \quad \varepsilon = \epsilon\sqrt{\mu} \text{ (wave steepness)}$$

- $\epsilon \sim \mu \ll 1$: Boussinesq (KdV) regime.

- $\epsilon \ll 1, \quad \frac{h^2}{\lambda_x^2} \sim \epsilon, \quad \frac{h^2}{\lambda_y^2} \sim \epsilon^2$: weakly transverse (KP) regime.

- $\epsilon \sim 1, \quad \mu \ll 1$: shallow water regime (Saint-Venant, Green-Naghdi).

- $\varepsilon \ll 1$: "full dispersion" regime.

- Zakharov-Craig-Sulem formulation of the water wave problem (posed in the fixed domain, $\mathbb{R}^d, d = 1, 2$), in dimensionless form :

$$\begin{cases} \partial_t \psi + \zeta + \frac{\epsilon}{2}|\nabla\psi|^2 - \frac{\epsilon}{2}(\frac{1}{\mu} + \epsilon^2|\nabla\zeta|^2)(Z_\mu(\epsilon\zeta)\psi)^2 = 0 \\ \partial_t \zeta + \epsilon\nabla\psi \cdot \nabla\zeta - (\frac{1}{\mu} + \epsilon^2|\nabla\zeta|^2)Z_\mu(\epsilon\zeta)\psi = 0, \end{cases} \tag{1}$$

where ζ is the elevation of the wave, ψ the trace of the velocity potential at the free surface and $Z_\mu(\epsilon\zeta)$ is the (nonlocal) Dirichlet to Neumann operator

The modulation (Schrödinger) regime

One introduces a fixed wave vector $\mathbf{k} \in \mathbb{R}^d, d = 1, 2$. Setting $\omega = \omega(\mathbf{k}) = \big(|\mathbf{k}| \tanh(\sqrt{\mu}|\mathbf{k}|)\big)^{1/2}$ the dispersion relation of surface gravity waves, one looks for an approximate solution of the water wave system on the form

$$U_{\text{app}}(X,t) = U_0(X,t) + \varepsilon U_1(X,t) + \varepsilon U^2(X,t), \tag{2}$$

where $X = (x,y)$ or x and U_0 is a sum of a modulated wave packet and of an induced mean mode ϕ

$$U_0(X,t) = \begin{pmatrix} i\omega\psi(\varepsilon X, \varepsilon t) \\ \psi(\varepsilon X, \varepsilon t) \end{pmatrix} e^{i(\mathbf{k}\cdot X - \omega t)} + \text{c.c.} + \begin{pmatrix} 0 \\ \phi(\varepsilon X, \varepsilon t) \end{pmatrix}. \tag{3}$$

- This leads to the Benney-Roskes system coupling ψ, ϕ and the leading term ζ of the surface elevation

$$\begin{cases} \partial_\tau \psi + \omega' \partial_x \psi - i\varepsilon \frac{1}{2}(\omega'' \partial_x^2 + \frac{\omega'}{|\mathbf{k}|}\partial_y^2)\psi \\ + \varepsilon i[|\mathbf{k}|\partial_x \phi + \frac{|\mathbf{k}|^2}{2\omega}(1-\sigma^2)\zeta + 2\frac{|\mathbf{k}|^4}{\omega}(1-\alpha)|\psi|^2]\psi = 0 \\ \partial_\tau \zeta + \sqrt{\mu}\Delta\phi = -2\omega|\mathbf{k}|\partial_x|\psi|^2, \\ \partial_\tau \phi + \zeta = -|\mathbf{k}|^2(1-\sigma^2)|\psi|^2. \end{cases} \quad (4)$$

- A similar system has been derived by Zakharov and Rubenchik (1972) as a "universal" Hamiltonian system describing the interaction of short and long waves.

One can derived a simplified system from the Benney-Roskes system using the fact that at leading order, ψ travels at the group velocity $c_g = \nabla\omega(\mathbf{k})$. This amounts in particular in replacing ∂_τ in the two last equations in the Benney-Roskes system by $-c_g\nabla$, where ∇ denotes here the gradient with respect to the variable $x - c_g t$.
This leads to the Davey-Stewartson system :

$$\begin{cases} \partial_\tau\psi - \frac{i}{2}\left(\omega''\partial_x^2 + \frac{\omega'}{|\mathbf{k}|}\partial_y^2\right)\psi + i(\beta\partial_x\phi + 2\frac{|\mathbf{k}|^4}{\omega}(1-\tilde{\alpha}|\psi|^2)\psi = 0 \\ [(\sqrt{\mu} - {\omega'}^2)\partial_x^2 + \sqrt{\mu}\partial_y^2]\phi = -2\omega\beta\partial_x|\psi|^2, \end{cases} \tag{5}$$

where

$$\beta = |\mathbf{k}|(1 + (1-\sigma^2)\frac{\omega'|\mathbf{k}|}{2\omega}, \quad \tilde{\alpha} = \frac{1}{4}(1-\sigma^2)^2,$$

α as previously while ζ is given by

$$\zeta = \tilde{\omega}'(|\mathbf{k}|)\partial_x\phi - |\mathbf{k}|^2(1-\sigma^2)|\psi|^2.$$

- In the infinite depth case the Davey-Stewartson system reduces to the "hyperbolic" cubic nonlinear Schrödinger equation derived in 1968 by Zakharov :

$$i\psi_t + \psi_{xx} - \psi_{yy} + |\psi|^2\psi = 0 \tag{6}$$

The complete *rigorous* justification of an asymptotic system, say of water waves, involves four main steps.

- Formal derivation of the models and proof of the *consistency* of the asymptotic systems with the full Euler system with free surface. This is not a dynamical issue (this reduces to approximate a nonlocal operator by a "simpler" (often local) one.
- The proof of the well-posedness of the Euler system on the correct time scales $1/\epsilon$. This difficult step has been achieved in Craig (1986) in the KdV regime and by Alvarez-Samaniego, Lannes (2008) in most of relevant regimes in absence of surface tension (see M. Ming-P. Zhang-Z. Zhang 2011 for gravity-capillary waves in the weakly transverse regime).
- Establishing long time, (that is on time scales of order at least $1/\epsilon$), for instance existence of solutions to the Boussinesq systems (see below) satisfying uniform bounds with respect to ϵ. Can be not easy for systems, such as Boussinesq (see Li Xu-JCS 2012).
- Assuming the previous steps, proving the optimal error estimates (for instance $O(\epsilon^2 t)$ in the KdV -Boussinesq regime) (see W. Craig 1986 for KdV, Bona-Colin-Lannes 2005 for the Boussinesq systems and Lannes 2013 AMS book for many other situations).

- Cf the analogy with the theory of finite difference schemes : a finite different scheme is convergent if and only if it is consistant and stable.

Some 1D examples

The KdV equation

$$\begin{cases} u_t - 6uu_x + u_{xxx} = 0, \\ u(x,0) = u_0(x), \ x \in \mathbb{R}. \end{cases} \tag{7}$$

Spectral problem for the Schrödinger operator

$$-\frac{d^2\psi}{dx^2} + u(\cdot, t)\psi.$$

- Gardner-Greene-Kruskal-Miura (1967)
- Lax (1968)
- In the context of water waves KdV writes

$$u_t + u_x + \epsilon uu_x + \epsilon uu_x = 0.$$

Decomposition in solitons

- Rigorous results by P.C. Schuur (1986), when u_0 is sufficiently smooth and decays sufficiently rapidly for $|x| \to \infty$:
- In absence of solitons,

$$\sup_{x \geq -t^{1/3}} |u(x,t)| = O(t^{-2/3}), \quad \text{as } t \to \infty.$$

- General case :

$$\sup_{x \geq -t^{1/3}} |u(x,t) - u_d(x,t)| = O(t^{-1/3}), \quad \text{as } t \to \infty.$$

When one does not restrict to waves going in one direction, one get systems that are not integrable, even in one space dimension.
For instance the following Boussinesq systems for surface water waves (obtained in the "KdV" regime) are not integrable

$$\begin{cases} \partial_t \eta + \operatorname{div} \mathbf{v} + \epsilon \operatorname{div}(\eta \mathbf{v}) + \epsilon(a \operatorname{div}\Delta \mathbf{v} - b\Delta \eta_t) = 0 \\ \partial_t \mathbf{v} + \nabla \eta + \epsilon \frac{1}{2}\nabla(|\mathbf{v}|^2) + \epsilon(c\nabla\Delta\eta - d\Delta \mathbf{v}_t) = 0 \end{cases}, \quad (x_1, x_2) \in \mathbb{R}^2,\ t \in \mathbb{R}. \tag{8}$$

where a, b, c, d are modelling constants satisfying the constraint $a + b + c + d = \frac{1}{3}$ (or $\frac{1}{3} - \tau$ for gravity-capillary waves) and ad hoc conditions implying that the well-posedness of linearized system at the trivial solution $(0, \mathbf{0})$.
The linear wave equation was formally derived in this context by Lagrange....)

Generalities on the Cauchy problem by PDE techniques

$$\begin{cases} \partial_t u = u'(t) = iLu(t) + F(u(t)), \\ u(0) = u_0. \end{cases} \tag{9}$$

Here $u = u(x,t)$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}$. L is a skew-adjoint operator defined in Fourier variables by

$$\widehat{Lf}(\xi) = p(\xi)\hat{f}(\xi),$$

where the symbol p is a real function (not necessary a polynomial). F is a nonlinear term depending on u and possibly on its space derivatives. The linear part of (9) thus generates a unitary group $S(t)$ in $L^2(\mathbb{R}^n)$ (and in all Sobolev spaces) which is unitary equivalent to $\hat{u}_0 \mapsto e^{itp(\xi)}\hat{u}_0$.

A natural way to prove LWP, inspired from the ODE case is to try to implement a Picard iterative scheme on the integral *Duhamel* formulation of (9), that is

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))ds. \tag{10}$$

where as already mentioned $S(t)$ denotes the unitary group in $L^2(\mathbb{R}^n)$ generated by iL.
We are thus reduced to finding a functional space $X_\tau \subset C([-\tau, +\tau]; H^s(\mathbb{R}^n))$, $\tau > 0$, such that for any bounded $B \subset H^s(\mathbb{R}^n)$, there exists $T > 0$ such that for any $u_0 \in B$, the right hand side of (10) is a contraction in a suitable ball of X_T.

- It is only in very special situations that the choice $X_\tau = C([-\tau, +\tau]; H^s(\mathbb{R}^n))$ is possible, for instance when F is lipschitz if $s = 0$, or in the case of NLS when $n = 1$ and $s > 1/2$ (exercice !).
- So, in this approach, the crux of the matter is the choice of an appropriate space X_τ. This can be carried out by using various dispersive estimates or by using a *Bourgain type* space.
- This method has the big advantage (on a compactness one that we will describe below for instance) of providing "for free" the uniqueness of the solution, the strong continuity in time and the "smoothness" of the flow (actually the only limitation of the smoothness of the flow is that of the smoothness of the nonlinearity).
- One should not forget that all this (very often hard) work provides merely a "Cauchy-Lipschitz" type result, and in particular does not give any insight on the (long time) dynamics...

The compactness method.

The rough idea is to construct approximate solutions by regularizing the equation, the data or the unknown (for instance by truncating high frequencies) and then to get a priori bounds on those approximate solutions. The fact that closed balls in infinite dimensional normed spaces are not relatively compact gives serious trouble.

This method does not provide neither the uniqueness of solutions nor the strong continuity in time or the continuity of the flow map.

- Of course PDE techniques go beyond the local well-posedness issues and can provide insights on scattering of small solutions, finite time blow-up, existence, qualitative and stability properties of "localized" (solitary waves) solutions,....

PDE techniques for KdV

- No decomposition in solitary waves + dispersion, but very partial results : orbital and asymptotic stability of the solitary wave, "multi-soliton" in the non integrable case,... (Martel-Merle).
- Resolution of the global Cauchy problem in large Sobolev spaces ($H^s(\mathbb{R}), s > -\frac{3}{4}$) (Bourgain,Kenig-Ponce -Vega).
- Apply of course to non-integrable equations.
- The fancy local well-posedness results are useless in the justification program. Here one just need a standard well -posedness result for KdV, but they are needed to get global results eg in the energy space $H^1(\mathbb{R})$.

The nonlinear Schrödinger equation (Gross-Pitaevskii)

- Zakharov-Shabat (1972). We consider the defocusing case, with the "correct" boundary condition (allowing "dark" and "black" solitons). The physical context is here nonlinear optics or quantum fluids.

$$i\partial_t \Psi = \partial_{xx}\Psi + \Psi(1 - |\Psi|^2) \quad \text{in } \mathbb{R} \times \mathbb{R}, \tag{11}$$

with a finite **Ginzburg-Landau energy**, namely

$$\mathcal{E}(\Psi) = \frac{1}{2}\int_{\mathbb{R}} |\nabla \Psi|^2 + \frac{1}{4}\int_{\mathbb{R}} (1 - |\Psi|^2)^2 \equiv \int_{\mathbb{R}} e(\Psi).$$

$$E = \{v \in H^1_{\text{loc}}(\mathbb{R}), \ \text{ s.t. } \mathcal{E}(v) < +\infty\},$$

(Natural energy space)

- **Inverse scattering**

$$i\partial_t u + \partial_x^2 u + (1 - |u|^2)u = 0 \tag{12}$$

Zakharov-Shabat (1973) consider the case when $|u(x,t)|^2 \to c > 0$, $|x| \to \infty$ (propagation of waves through a condensate of constant density).
More precisely, the GP has a Lax pair (B_u, L_u), where (for $c = 1$)

$$L_u = i \begin{pmatrix} 1+\sqrt{3} & 0 \\ 0 & 1-\sqrt{3} \end{pmatrix} \partial_x + \begin{pmatrix} 0 & \bar{u} \\ u & 0 \end{pmatrix} \tag{13}$$

$$B_u = -\sqrt{3} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \partial_x^2 + \begin{pmatrix} \frac{|u|^2-1}{\sqrt{3}+1} & i\partial_x \bar{u} \\ -i\partial_x \bar{u} & \frac{|u|^2-1}{\sqrt{3}-1} \end{pmatrix} \tag{14}$$

So that u satisfies (GP) if and only if

$$\frac{d}{dt} L_u = [L_u, B_u] \tag{15}$$

- As a consequence, the 1 D Gross-Pitaevskii equation has an infinite number of (formally) conserved energies $E_k, k \in \mathbb{N}$. and momentum $P_k, k \in \mathbb{N}$.

For instance,

$$I_5 = \int_{\mathbb{R}} \{2|u|^6 + 6|u|^2|u_x|^2 + (\frac{d}{dx}|u|^2)^2 + |u_{xx}|^2 - 2\}.$$

It is of course necessary to prove rigorously that the E_k and the P_k are well defined and conserved by the GP flow, in a suitable functional setting.

- Justified in Bethuel-Gravejat-S-Smets (2009) and used to study the long wave (KdV) limit of GP.

Rigorous results via IST

- I do not know of any rigorous result on the decomposition in solitary waves.

Solitary waves :

(GP) has two types of traveling waves :

- The **"dark"** solitons : $\mathbf{v}_c(x) = \sqrt{\frac{2-c^2}{2}} \tanh\left(\frac{\sqrt{2-c^2}}{2}x\right) - i\frac{c}{\sqrt{2}}$.
- The **"black"** soliton :

$$\mathbf{v}_0(x) = \tanh\left(\frac{x}{\sqrt{2}}\right).$$

- Note that when $0 < c < \sqrt{2}$, $\mathbf{v}_c(x) \neq 0, \forall x$.
- **Orbital stability of the black soliton** :
  Béthuel-Gravejat-Saut-Smets (Indiana Math. J. 2008).
  P. Gérard and Zhifei Zhang (2008) for a different result by Inverse Scattering techniques.

The Cauchy problem by PDE methods

- Zhidkov 1987 : GWP in the "Zhidkov space" $Z^1 = \{u \in L^\infty(\mathbb{R}); u_x \in L^2(\mathbb{R})\}$.
- P. Gérard 2006-2008 : GWP in the *energy space* (also in higher dimensions). See also C. Gallo (2008).
- The 2D Gross-Pitaevskii equation

$$i\psi_t + \Delta\psi + \psi(1 - |\psi|^2) = 0 \tag{16}$$

is not integrable but has a very rich dynamics (Béthuel, Gravejat, Smets, JCS, Chiron, Maris, Gallo, Gustafson, Nakanishi, Tsaï,...).

The Benjamin-Ono equation

$$u_t + uu_x - \mathcal{H}u_{xx} = 0, \tag{17}$$

where $\mathcal{H}$ is the Hilbert transform.

- A (not too good) model for internal waves.
- Formally integrable by IST (Ablowitz-Fokas 1983) associated to a nonlocal Riemann-Hilbert problem.
- Rigorous result by Coifman and Wickerhauser 1990 : GWP for small initial data.

The Benjamin-Ono equation by PDE techniques

- The BO equation is quasilinear (Molinet-JCS -Tzvetkov 2001)

Theorem

Let $s \in \mathbb{R}$ and T be a positive real number. Let $S(t) = e^{t\mathcal{H}\partial_x}$. Then there does not exist a space X_T continuously embedded in $C([-T,T], H^s(\mathbb{R}))$ such that there exists $C > 0$ with

$$\|S(t)\phi\|_{X_T} \leq C\|\phi\|_{H^s(\mathbb{R})}, \quad \phi \in H^s(\mathbb{R}), \tag{18}$$

and

$$\left\| \int_0^t S(t-t') \left[u(t')u_x(t')\right] dt' \right\|_{X_T} \leq C\|u\|^2_{X_T}, \quad u \in X_T. \tag{19}$$

Note that the two conditions would be needed to implement a Picard iterative scheme in the space X_T. As a consequence one has the following result.

Theorem

Fix $s \in \mathbb{R}$. Then there does not exist a $T > 0$ such that BO admits a unique local solution defined on the interval $[-T, T]$ and such that the flow-map data-solution

$$\phi \longmapsto u(t), \quad t \in [-T, T],$$

is C^2 differentiable at zero from $H^s(\mathbb{R})$ to $H^s(\mathbb{R})$.

- Bad small/large frequencies interactions.
- The flow cannot be C^1 for data in $H^s, s \geq 0$, Koch-Tzvetkov (2005) (a typical property of quasilinear hyperbolic equations).

- The "fractional KdV equation"

$$u_t + uu_x + D^\alpha u_x = 0, \quad \widehat{D^\alpha f}(\xi = |\xi|^\alpha \hat{f}(\xi),$$

  is quasilinear for $\alpha < 2$ and semilinear when $\alpha \geq 2$.
- The (integrable) ILW equation (replace the Fourier symbol $-i\text{sgn}\xi$ of the Hilbert transform by $i(\xi\coth\xi - 1)$) is also quasilinear.

BO by PDE methods

- LWP below $H^{3/2}(\mathbb{R})$ was not easy ! Koch-Tzvetkov (2003), $H^s(\mathbb{R}), s > 5/4$. Kenig-Koenig (2003), $H^s(\mathbb{R}), s > 9/8$.
- Breakthrough by T. Tao via a gauge transform, GWP in $H^1(\mathbb{R})$, (2003).
- GWP in the energy space $H^{1/2}(\mathbb{R})$, Burq-Planchon (2007), Ionescu-Kenig (2007).
- Of course those results do not provide any qualitative properties of solutions.
- Orbital stability of the soliton (Albert, Bona,..).
- Uniqueness of the solitary wave (Amick, Toland).
- Decomposition in solitons ?

The KP equations

Where does KP come from ?

We recall here how the KP equations were first derived. The (classical) Kadomtsev-Petviashvili (KP) equations

$$(u_t + u_{xxx} + uu_x)_x \pm u_{yy} = 0 \tag{20}$$

were introduced (1970) to study the transverse stability of the solitary wave solution of the Korteweg- de Vries equation which reads in the theory of water-waves (dropping the small parameter ϵ)

$$u_t + u_x + uu_x + (\frac{1}{3} - T)u_{xxx} = 0, \; x \in \mathbb{R}, \; t \geq 0. \tag{21}$$

Here $T \geq 0$ is the Bond number which measures surface tension effects in the context of surface hydrodynamical waves.

Actually the (formal) analysis in KP 1970 consists in looking for a *weakly transverse* perturbation of the one-dimensional transport equation

$$u_t + u_x = 0. \tag{22}$$

This perturbation is obtained by a Taylor expansion of the dispersion relation $\omega(k_1, k_2) = \sqrt{k_1^2 + k_2^2}$ of the two-dimensional linear wave equation assuming $|k_1| \ll 1$ and $\frac{|k_2|}{|k_1|} \ll 1$.
Namely, one writes formally

$$\omega(k_1, k_2) \sim \pm k_1 \left(1 + \frac{k_2^2}{k_1}\right)$$

which, with the $+$ sign say, amounts, coming back to the physical variables, to adding a nonlocal term to the transport equation,

$$u_t + u_x + \frac{1}{2}\partial_x^{-1} u_{yy} = 0. \tag{23}$$

Here the operator ∂_x^{-1} is defined via Fourier transform,

$$\widehat{\partial_x^{-1} f}(\xi) = \frac{i}{\xi_1}\widehat{f}(\xi), \text{ where } \xi = (\xi_1, \xi_2).$$

The same formal procedure is applied in KP 1970 to the KdV equation (21), assuming that the transverse dispersive effects are of the same order as the x-dispersive and nonlinear terms, yielding the KP equation in the form

$$u_t + u_x + uu_x + (\frac{1}{3} - T)u_{xxx} + \frac{1}{2}\partial_x^{-1}u_{yy} = 0. \qquad (24)$$

By change of frame and scaling, (24) reduces to (20) with the $+$ sign (KP II) when $T < \frac{1}{3}$ and the $-$ sign (KP I) when $T > \frac{1}{3}$.[1]
Note that KP I has a focusing character while KP II has a defocusing one.

1. The same formal procedure could be applied to *any* one-dimensional dispersive equation leading to its quasi one-dimensional extension. It suffices to add the nonlocal term $\partial_x^{-1}u_{yy}$.

Note however that $T > \frac{1}{3}$ corresponds to a layer of fluid of depth smaller than 0.46 cm, and in this situation viscous effects due to the boundary layer at the bottom cannot be ignored. One could then say that the KP I equation does not exist in the context of water waves, but it appears naturally in other contexts, for instance as the transonic limit of the 2D Gross-Pitaevskii equation (Béthuel-Gravejat-JCS 2008, Chiron-Rousset 2010, Chiron-Maris 2012).

- One can justify rigorously the KP approximation of the water waves system (see David Lannes book "Water waves : mathematical theory and asymptotics", AMS 2013, and also for the other asymptotic regimes).
- A drawback of the KP approximation is the poor error estimate which makes KP a "bad" asymptotic model.

  In fact, see D. Lannes (2002), D. Lannes -JCS (2006), one gets

$$||U_{\text{Euler}} - U_{KP}|| = o(1), (O(\sqrt{\epsilon}) \text{ with some additional constraint})$$

  instead of $O(\epsilon^2 t)$ which should be the optimal error estimate (achieved in the isotropic Boussinesq regime).
- This is due to the very bad approximation of the water waves dispersion in the neighborhood of the frequency $\xi_1 = 0$.

Weakly transverse Boussinesq systems

One can however derive in the KP scaling a five parameters family of weakly transverse Boussinesq systems which are consistent with the Euler system, do not suffer from the unphysical zero-mass constraint (due to the artificial singularity at $\xi_1 = 0$) and have the same precision $O(\epsilon^2 t)$ as the isotropic ones (see D. Lannes-JCS 2006) :

$$\begin{cases} \partial_t v + \partial_x \zeta + \epsilon(a\partial_x^3 \zeta - b\partial_x^2 \partial_t v + v\partial_x v + \frac{1}{2} w\partial_x w) + \frac{1}{2}\epsilon^{3/2} w\partial_y w = 0 \\ \partial_t w + \sqrt{\epsilon}\partial_y \zeta + \epsilon(-e\partial_x^2 \partial_t w + w\partial_y w + \frac{1}{2} v\partial_x w) + \epsilon^{3/2}(f\partial_x^2 \partial_y \zeta + \frac{1}{2} v\partial_y v) = 0 \\ \partial \zeta + \partial_x v + \sqrt{\epsilon}\partial_y w + \epsilon(v\partial_x \zeta + \zeta\partial_x v + c\partial_x^3 v - d\partial_x^2 \partial_t \zeta) \\ \quad + \epsilon^{3/2}(w\partial_y \zeta + \zeta \partial_y w + g\partial_x^2 \partial_y w) = 0, \end{cases} \tag{25}$$

- Unfortunately none of those systems is known to be integrable by IST as actually the Boussinesq systems describing waves going in two directions in the Boussinesq (KdV) regime...

The constraint problem

To give sense to the operator $\partial_x^{-1}\partial_y^2$ imposes a constraint on the solution u of KP, which, in some sense, has to be an x-derivative. This is achieved, for instance, if $u \in \mathcal{S}'(\mathbb{R}^2)$ is such that

$$\xi_1^{-1}\,\xi_2^2\,\widehat{u}(t,\xi_1,\xi_2) \in \mathcal{S}'(\mathbb{R}^2)\,, \qquad (26)$$

thus in particular if $\xi_1^{-1}\,\widehat{u}(t,\xi_1,\xi_2) \in \mathcal{S}'(\mathbb{R}^2)$. Another possibility to fulfill the constraint is to write u as

$$u(t,x,y) = \frac{\partial}{\partial x} v(t,x,y), \qquad (27)$$

where v is a continuous function having a classical derivative with respect to x, which, for any fixed y and $t \neq 0$, vanishes when $x \to \pm\infty$. Thus one has

$$\int_{-\infty}^{\infty} u(t,x,y)dx = 0, \qquad y \in \mathbb{R}, \quad t \neq 0, \qquad (28)$$

Of course the differentiated version, namely

$$(u_t + u_x + uu_x + u_{xxx})_x \pm \partial_y^2 u = 0, \tag{29}$$

can make sense without any constraint on u, and so does the Duhamel integral representation,

$$u(t) = S(t)u_0 - \int_0^t S(t-s)(u(s)u_x(s))ds, \tag{30}$$

where $S(t)$ denotes the (unitary in all Sobolev spaces $H^s(\mathbb{R}^2)$) KP group,

$$S(t) = e^{-t(\partial_x - \partial_{xxx} \pm \partial_x^{-1}\partial_y^2)} \,. \tag{31}$$

- The constraint is needed to define the Hamiltonian

$$\frac{1}{2}\int \left[u_x^2 \pm (\partial_x^{-1} u_y)^2 - \frac{u^3}{3}\right], \quad (+ \text{ corresponds to } KPI). \tag{32}$$

- The constraint is linked to the introduction of ∂_x^{-1} leading to an artificial singularity on the dispersion at $\xi_1 = 0$. As previously mentioned, this yields poor error estimates with the solution of the original problem, *eg* the water wave equations, (see Lannes 2003) and makes KP a rather poor asymptotic system...

- The following theorem (Molinet -JCS-Tzvetkov 2007) holds in fact for a class of KP equations with general dispersion in x.

Theorem

Let $\varphi \in L^1(\mathbb{R}^2) \cap H^{2,0}(\mathbb{R}^2)$ and

$$u \in C([0,T]\,;\, H^{2,0}(\mathbb{R}^2)) \tag{33}$$

be a distributional solution of KP. Then, for every $t \in (0,T]$, $u(t,\cdot,\cdot)$ is a continuous function of x and y which satisfies

$$\int_{-\infty}^{\infty} u(t,x,y)dx = 0 \quad \forall y \in \mathbb{R},\ \ \forall t \in (0,T]$$

in the sense of generalized Riemann integrals. Moreover, $u(t,x,y)$ is the derivative with respect to x of a C^1_x continuous function which vanishes as $x \to \pm\infty$ for every fixed $y \in \mathbb{R}$ and $t \in [0,T]$.

Rigorous results on the Cauchy problem by IST

- Formal results by Zakharov-Manakov (1979), Manakov (1981), Ablowitz-Fokas (1983).
- The KP II is linked to the spectral problem for the heat equation

$$\psi_y - \psi_{xx} - u(.,.,t)\psi = 0.$$

- The KP I is linked to the spectral problem for the Schrödinger equation

$$i\psi_y + \psi_{xx} + u(.,.,t)\psi = 0$$

The KP-II case

- Wickerhauser (1987) : GWP for initial data having 7 small derivatives in $L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$.
- The smallness condition is for the forward spectral problem.

The KP-II case

- Zhou (1990) : GWP for small initial data in a suitable space.
- The smallness condition is for the forward spectral problem.
- Asymptotics : see below.

KP by PDE methods

- The two equations are of very different nature. KP II is *semilinear*, while KP I is *quasilinear* and this makes a huge difference in the treatment of the Cauchy problem.

- KP II is *semilinear* in the sense that the Cauchy problem can be solved by Picard iteration on the Duhamel formulation yielding a smooth (at least C^1!) flow map $u(0) \mapsto u(t)$.
- Bourgain (1993) : KPII is locally (thus globally) well-posed for data in $L^2(\mathbb{R}^2)$ (and also $L^2(\mathbb{T}^2)$).
- Takaoka-Tzvetkov (2001), LWP for data in $H^{s_1,s_2}(\mathbb{R}^2)$, $s_1 > -\frac{1}{3}, s_2 \geq 0$.

KPI is quasilinear

Theorem

(Molinet-S-Tzvetkov 2002).
Let $(s_1, s_2) \in \mathbb{R}^2$ *(resp.* $s \in \mathbb{R}$*). Then there exists no* $T > 0$ *such that KPI admits a unique local solution defined on the interval* $[-T, T]$ *and such that the flow-map*

$$S_t : u(0) \longmapsto u(t), \quad t \in [-T, T]$$

is C^2 *differentiable at zero from* $H^{s_1,s_2}(\mathbb{R}^2)$ *to* $H^{s_1,s_2}(\mathbb{R}^2)$*, (resp. from* $H^s(\mathbb{R}^2)$ *to* $H^s(\mathbb{R}^2)$*).*

Remark

- *As in the case of the Benjamin-Ono equation, the previous result implies that one cannot solve the KPI equation by iteration on the Duhamel formulation for data in Sobolev spaces $H^{s_1,s_2}(\mathbb{R}^2)$ or $H^s(\mathbb{R}^2)$, for any value of s, s_1, s_2. This is in contrast with the KP II equation.*
- *The reason is the large set of zeroes of a "resonant function".*

Let

$$\sigma(\tau,\xi,\eta) = \tau - \xi^3 - \frac{\eta^2}{\xi},$$

$$\sigma_1(\tau_1,\xi_1,\eta_1) = \sigma(\tau_1,\xi_1,\eta_1),$$

$$\sigma_2(\tau_1,\xi,\eta_1,\tau_1,\xi_1,\eta_1) = \sigma(\tau-\tau_1,\xi-\xi_1,\eta-\eta_1).$$

We then define

$$\chi(\xi,\xi_1,\eta,\eta_1) := 3\xi\xi_1(\xi-\xi_1) - \frac{(\eta\xi_1-\eta_1\xi)^2}{\xi\xi_1(\xi-\xi_1)}.$$

Note that $\chi(\xi,\xi_1,\eta,\eta_1) = \sigma_1+\sigma_2-\sigma$. The "resonant" function $\chi(\xi,\xi_1,\eta,\eta_1)$ plays an important role in the analysis. The “"large" set of zeros of $\chi(\xi,\xi_1,\eta,\eta_1)$ is responsible for the ill-posedness issues.

In contrast, the corresponding resonant function for the KP II equation is

$$\chi(\xi, \xi_1, \eta, \eta_1) := 3\xi\xi_1(\xi - \xi_1) + \frac{(\eta\xi_1 - \eta_1\xi)^2}{\xi\xi_1(\xi - \xi_1)}.$$

Since it is essentially the sum of two squares, its set of zeroes is small and this is the key point to establish the crucial bilinear estimate in Bourgain' s method

The bad structure of the resonant set for KPI was also the obstruction faced by Zakharov to develop for KP I his theory of Birkhoff normal form for KP II.

Global well-posedness of KPI (Molinet-S-Tzvetkov 2001)

$$||\phi||_Z = |\phi|_2 + |\phi_{xxx}|_2 + |\phi_y|_2 + |\phi_{xy}|_2 + |\partial_x^{-1}\phi_y|_2 + |\partial_x^{-2}\phi_{yy}|_2.$$

Theorem

Let $\phi \in Z$. Then there exists a unique global solution u of the KP-I equation with initial data ϕ, such that $u \in L^\infty_{loc}(\mathbb{R}_+; Z)$, $u_t \in L^\infty_{loc}(\mathbb{R}_+; H^{-1}(\mathbb{R}^2))$ [2]*.*
Moreover $M(u(t)) = M(\phi)$, $E(u(t)) = E(\phi)$, and $u,\ u_y,\ u_{xx} \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^2))$. In particular $u \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^2)$.

2. In particular $u \in C_w(\mathbb{R}_+; Z)$.

- Use a compactness methods and some invariants of KPI (technical problems to justify them !)
- Strange fact : the KP invariants do not make sense after a certain rank.
  For instance, the invariant which should control $||u_{xxx}(., t)||_{L^2}$ contains the L^2 norm of $\partial_x^{-1}\partial_y(u^2)$ which does not make sense for a non zero function u in the Sobolev space $H^3(\mathbb{R}^2)$. Actually, one checks easily that if $\partial_x^{-1}\partial_y(u^2) \in L^2(\mathbb{R}^2)$, then $\int_{\mathbb{R}^2} \partial_y(u^2)dx = \partial_y \int_{\mathbb{R}^2} u^2 dx \equiv 0, \ \forall y \in \mathbb{R}$, which, with $u \in L^2(\mathbb{R}^2)$, implies that $u \equiv 0$. Similar obstructions occur for the higher order "invariants".
- Use instead a quasi invariant.
- Ionescu-Kenig-Tataru 2008 : GWP in the energy space.

- GWP on a background of a non localized solution (e.g. the KdV solitary wave) for KPI and KPII (Molinet-JCS-Tzvetkov 2007-2011).
- Nonlinear transverse stability (resp instability) for the KdV soliton embedded in the KPII (resp KP I) equation (Rousset-Tzvetkov 2009-2011).
- The nature of the instability for KPI is unknown. Numerical simulations (Klein-S 2012) suggest a breaking of symmetry towards lumps like or Zaitsev solitons.
- The nonlinear instability for KPI has been established by Zakharov using IST but Rousset-Tzvetkov techniques are general and apply to other (non integrable) equations.

This could be described by KP II

- Another strange fact
  The KP I has an explicit soliton solution, the lump (Manakov,-Zakharov- Bordag -Matveev 1977).

$$\phi_c(x-ct,y) = \frac{8c(1-\frac{c}{3}(x-ct)^2+\frac{c^2}{3}y^2)}{[1+\frac{c}{3}(x-ct)^2+\frac{c^2}{3}y^2]^2}. \tag{34}$$

- A rigorous theory of stability of the lump is unknown (to my knowledge).
- One does not whether or not the lump is a ground state (see below).

- On the other hand generalized KP I equations (uu_x changed in $u^p u_x$,) have ground states solutions, defined by a variational method (de Bouard-S 1997) ;

$$E_{KP}(\psi) = \frac{1}{2}\int_{\mathbb{R}^2}(\partial_x\psi)^2 + \frac{1}{2}\int_{\mathbb{R}^2}(\partial_x^{-1}\partial_y\psi)^2 - \frac{1}{2(p+2)}\int_{\mathbb{R}^2}\psi^{p+2},$$

and we define the action

$$S(N) = E_{KP}(N) + \frac{c}{2}\int_{\mathbb{R}^2} N^2.$$

We term *ground state*, a solitary wave N which minimizes the action S among all finite energy non-constant solitary waves of speed c.

It was proven in (deB-S) that ground states exist if and only if $c > 0$ and $1 \leq p < 4$. Moreover, when $1 \leq p < \frac{4}{3}$, the ground states are minimizers of the Hamiltonian E_{KP} with prescribed mass (L^2 norm). They are then orbitally stable.

- The uniqueness (up to the obvious symmetries), of the ground states is unknown. Also one does know whether or not the lump is a ground state (but all solitary waves cannot decay faster than $1/r^2$ (deB-S 1997) and moreover they have the same asymptotics as the lump, Gravejat 2008).
- As for BO solitary waves, the slow decay of KP solitary waves is due to the non smoothness of the dispersion symbol.

Asymptotics of small solutions to KP

- KP II : precise asymptotics (Kiselev 2001). The global decay is $0(1/t)$ but the precise asymptotics is different in different directions, described by the variable $\xi = x/t$ and $\eta = y/t$.
- KP I : formal analysis (Manakov-Santini-Takchtadzhyan 1980).
- For the generalized KP I/II equation

$$u_t + u^p u_x + u_{xxx} \pm \partial_x^{-1} u_{yy} = 0, \tag{35}$$

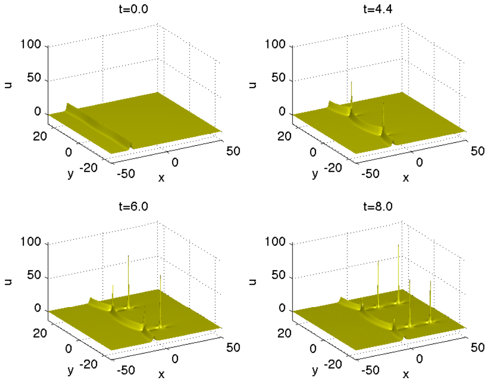
when $p > 1$, decay of the sup norm of small solutions as $O(1/t)$. (Hayashi-Naumkin-JCS 1999).

- As for other scalar equations (KdV, BO,..) the fancy LWP results for KP are not necessary in the justification program. One just need an easy LWP in $H^s(\mathbb{R}^2)$, s large enough. On the other hand they are crucial to justify the mass, energy,.. conservation and the orbital stability of KP I ground states.

Open questions around KP

- What is the meaning of the "KP hierarchy" since it is the sequence of Hamiltonian flows for Hamiltonian which do not make sense after a certain rank...
- Are the smallness conditions on the initial data imposed for IST techniques really necessary ?
- Is the explicit lump of KP I a ground state in the previous sense.
- Are the ground states unique (up to trivial symmetries).
- Groves and Sun (ARMA 2008) have proven the existence of "lump like" solutions to the capillary-gravity waves system.

- As far as the global well-posedness of the Cauchy problem is concerned, PDE methods win against IST techniques since they yield global well-posedness of arbitrary large solutions. On the other hand they do not information on the large time behavior of solutions. For instance, in the defocussing KP II case, one might decay of the L^∞ norm of all solution as $1/t$, as it is the case in the linear case (JCS 1993).
- Things are less clear for the focusing KP I. What is the dynamics for large t?
- Nature of the transverse instability of the KdV solitary wave (see next slide) ?



Transverse instability of the KdV solitary wave for KP I (Klein-S 2010)

The Davey-Stewartson systems

- The Davey-Stewartson-systems were historically derived from the water-waves system as an asymptotic model in the modulation regime.
- They are a limiting system of a more general one, the Benney-Roskes system in the theory of water waves, derived also by Zakharov and Rubenchik as a "universal system for the description of the interaction between short and long (acoustic type) waves".

The Benney-Roskes, Zakharov-Rubenchik system

$$\begin{cases} i\partial_t\psi = -\epsilon\partial_z^2\psi - \sigma_1\Delta_\perp\psi + (\sigma_2|\psi|^2 + W(\rho + D\partial_z\phi))\psi, \\ \partial_t\rho + \sigma_3\partial_z\phi = \Delta\phi - D\partial_z|\psi|^2, \\ \partial_t\phi + \sigma_3\partial_z\phi = -\dfrac{1}{M}\rho - |\psi|^2. \end{cases} \tag{36}$$

Here $\Delta_\perp = \partial_x^2 + \partial_y^2$ or ∂_x^2, $\Delta = \Delta_\perp + \partial_z^2$,$\sigma_1, \sigma_2, \sigma_3 = \pm 1$, $W > 0$ measures the coupling with acoustic type waves, $M \in (0, 1)$ is a Mach number, $D \in \mathbb{R}$ and $\epsilon \in \mathbb{R}$ is a nondimensional dispersion coefficient.

The Davey-Stewartson systems have in fact the general form

$$\begin{aligned} i\psi_t + a\partial_x^2\psi + b\partial_y^2\psi &= (\nu_1|\psi|^2 + \nu_2\partial_x\phi)\psi, \\ c\partial_x^2\phi + \partial_y^2\phi &= -\delta\partial_x|\psi|^2, \end{aligned} \tag{37}$$

where one can assume (up to a change of unknown) $b > 0$ and $\delta > 0$.

Classification : (37) is

$$\begin{aligned} \text{elliptic-elliptic if} \quad & (\operatorname{sgn} a, \operatorname{sgn} c) = (+1, +1), \\ \text{hyperbolic-elliptic if} \quad & (\operatorname{sgn} a, \operatorname{sgn} c) = (-1, +1), \\ \text{elliptic-hyperbolic if} \quad & (\operatorname{sgn} a, \operatorname{sgn} c) = (+1, -1), \\ \text{hyperbolic-hyperbolic if} \quad & (\operatorname{sgn} a, \operatorname{sgn} c) = (-1, -1). \end{aligned}$$

(it does not seem to occur in a physical situation).

- In the context of water waves, the equation for ϕ is hyperbolic when $\frac{C_g^2}{gh} > 1$, where $c_g = \frac{d\omega}{d\kappa}$, $\omega^2 = g\kappa(1+\kappa^2 T)\tanh \kappa h$, $\kappa = \sqrt{k^2+l^2}$, and where $T \geq 0$ is the surface tension coefficient.
- When $T = 0$ (purely gravity waves), one has $\frac{C_g^2}{gh} < 1$, and the corresponding DS systems are of type $(\pm, +)$.

The so-called *DS I and DS II systems* are integrable, but very particular cases of respectively elliptic-hyperbolic and hyperbolic-elliptic Davey-Stewartson systems. In fact they correspond to a very special choice of the coefficients in (37) obtained in the limit $kh \to 0$, $\epsilon = \kappa a \ll (kh)^2$ that has only a limited physical relevance.

The Davey-Stewartson II type systems

We consider here the Davey-Stewartson system that appears in deep water which we write for convenience as

$$\begin{cases} iu_t + u_{xx} - u_{yy} = \alpha|u|^2u + \beta u\phi_x, \\ \Delta\phi = \frac{\partial}{\partial x}|u|^2. \end{cases} \tag{38}$$

- The Davey-Stewartson system (38) is integrable by the Inverse Scattering method (see below for rigorous results) when

$$\alpha + \frac{\beta}{2} = 0.$$

  It is then known as the Davey-Stewartson II (DS II) system. The case $\beta < 0$ corresponds to the *defocusing* DS II, $\beta > 0$ to the *focusing* DS II. We will keep this terminology in the non integrable case.

Non existence of localized traveling waves (Ghidaglia-JCS 1996)

Theorem

The Davey-Stewartson system (38) can possess a non zero localized traveling wave solution only if

$$(i)\ \beta < 0, \quad \alpha \in (0, -\beta).$$

Moreover, (38) possesses a nontrivial **radial** *traveling wave solution if and only if*

$$(ii)\ \beta < 0, \quad \alpha + \frac{\beta}{2} = 0.$$

On the other hand Arkadiev-Pogrebkov-Polivanov (1989) have exhibited a family of explicit traveling waves (the lump solitons) for the focusing integrable DS II system having the profile($c = 0$) :

$$u_{\mathrm{lump}}(x, y, t) = \frac{2\bar{\nu}\exp\left(2i \text{ Im } (\lambda z) + 4i \text{ Re } (\lambda^2) t\right)}{|z + 4i\lambda t + \mu|^2 + |\nu|^2},$$

where $z = x + iy$ and λ, ν, μ are arbitrary complex constants. Note that the traveling wave profile $|u(x, y, 0)|$ is radial in appropriate variables.

- Numerical simulations (Klein-JCS 2013) suggest that the lump solution does not persists in the non integrable focusing DS II.

The Cauchy problem via PDE techniques

- Reduces to a "nonelliptic" NLS with a nonlocal term :

$$iu_t + u_{xx} - u_{yy} = \alpha |u|^2 u + uL(|u|^2),$$

where $\widehat{Lf}(\xi) = \frac{\xi_1^2}{|\xi|^2}\hat{f}(\xi)$.

- The Strichartz estimates are the same as for the usual Schrödinger group and one obtains the same results as for the cubic 2D NLS (Ghidaglia-JCS 1990) : LWP for initial data in $L^2(\mathbb{R}^2)$ or $H^1(\mathbb{R}^2)$ global for small L^2 data.

A blow-up property of the focusing DS II

- On the other hand, Ozawa (1992) by using a special case of the lump and a pseudo-conformal transform has constructed a solution of the Cauchy problem in the *focusing integrable* case whose L^2 norm blows up in finite time (the solution converges to a Dirac measure having as mass the L^2 norm of the initial data). The solution persists after the blow-up time and disperses as $t \to \infty$.
  This blow-up is carefully studied numerically in Klein-Roidot 2012 where its instability is suggested. On the other hand, the numerical simulations of Klein-S (2014) suggest that this blow-up does not persist in the non integrable case.

Blow-up in the non integrable focusing DS II ? (Klein-S 2014)

Write DS II as

$$i\epsilon\partial_t\psi + \epsilon^2\partial_{xx}\psi - \epsilon^2\partial_{yy}\psi + 2\rho\Delta^{-1}[(\partial_{yy} + (1-2\beta)\partial_{xx})\,|\psi|^2]\psi = 0, \quad (39)$$

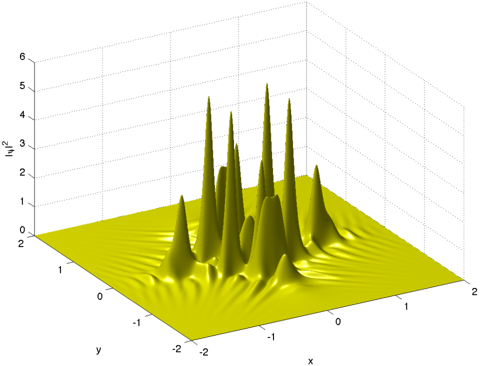
which involves the order zero nonlocal operator

$$\Delta^{-1}[(\partial_{yy} + (1-2\beta)\partial_{xx})].$$

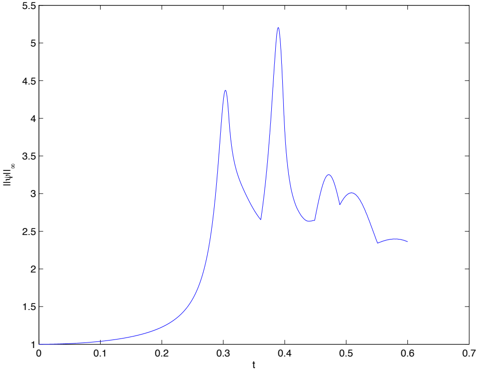
Note that the integrable case $\beta = 1$ is distinguished by the fact that the same hyperbolic operator appears in the linear and in the nonlinear part. In this case the equation is invariant under the transformation $x \to y$ and $\psi \to \bar{\psi}$ and (39) can be written in a "symmetric" form as

$$i\epsilon\partial_t\psi + \epsilon^2\Box\psi - 2\rho[(\Delta^{-1}\Box)|\psi|^2]\psi = 0, \quad (40)$$

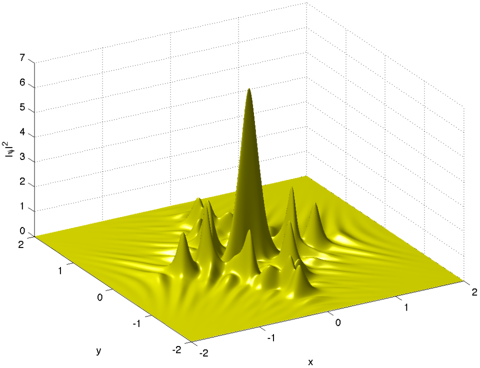
where $\Box = \partial_{xx} - \partial_{yy}$.



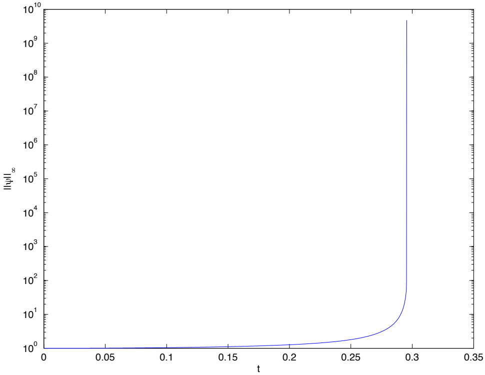
DS II, $\beta = 0.9, \epsilon = 0.1,\ t = 0.6,\ \psi_0 = \exp(-x^2 - y^2)$.



DS II, $\beta = 0.9, \epsilon = 0.1,\ t = 0.6,\ \psi_0 = \exp(-x^2 - y^2), ||\psi(\cdot, t)||_\infty$



DS II, $\beta = 1.1, \epsilon = 0.1,\ t = 0.6,\ \psi_0 = \exp(-x^2 - y^2)$.



DS II, $\beta = 1, \epsilon = 0.1,\ t = 0.6,\ \psi_0 = \exp(-x^2 - y^2), ||\psi(\cdot, t)||_\infty$

The appearance of blow-up is a very subtle and surprising phenomenon in DS II type systems. Recall that one does not expect blow-up in the cubic hyperbolic NLS equation

$$i\psi_t + \psi_{xx} - \psi_{yy} + |\psi|^2\psi = 0,$$

which does not possess any localized solitary waves (Ghidglia-JCS 1995).

Global results by IST

Write the integrable DS-II equation on the form

$$q_t = 2iq_{x_1x_2} + 16i[L(|q|^2)|q|, \quad q(\cdot,0) = q_0, \tag{41}$$

where L is defined as above and the $+$ sign corresponds to the focusing case, the $-$ sign to the defocusing case.

- Nice results by Beals-Coifman (1990), L-Y.Sung (1994-1995), Perry (2012).

Theorem

Sung (1995)
Let $q_0 \in \mathcal{S}(\mathbb{R}^2)$. Then DSII possesses a unique global solution u such that the mapping $t \mapsto q(\cdot, t)$ belongs to $C^\infty(\mathbb{R}, \mathcal{S}(\mathbb{R}^2))$ in the two cases :
(i) Defocusing.
(ii) Focusing and $|\widehat{q_0}|_1 |\widehat{q_0}|_\infty < \frac{\pi^3}{2} \left(\frac{\sqrt{5}-1}{2}\right)^2$.
Moreover, there exists $c_{q_0} > 0$ such that

$$|q(x,t)| \leq \frac{c_{q_0}}{|t|}, \quad (x,t) \in \mathbb{R}^2 \times \mathbb{R}^*.$$

- Note that such a result is unknown for the general *non integrable* DS-II systems, and also for the nonelliptic cubic NLS.

Remark

1. Sung obtains in fact the global well-posedness (without the decay rate) in the defocusing case under the assumption that $\hat{q}_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ *and* $q_0 \in L^p(\mathbb{R}^2)$ *for some* $p \in [1, 2)$.

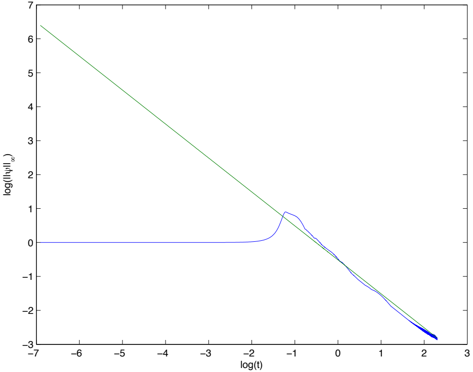
2. Recently, Perry (2012) has precised the asymptotics in the defocusing case for initial data in $H^{1,1}(\mathbb{R}^2) = \{f \in L^2(\mathbb{R}^2)$ *such that* $\nabla f, (1 + |\cdot|)f \in L^2(\mathbb{R}^2)\}$, *proving that the solution obeys the asymptotic behavior in the* $L^\infty(\mathbb{R}^2)$ *norm :*

$$q(x,t) = u(\cdot,t) + o(t^{-1}),$$

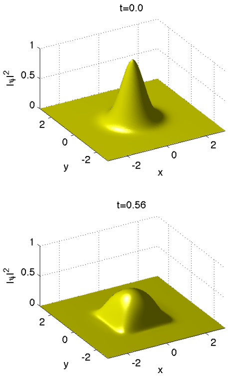
where u is the solution of the linearized problem, a result which is out of reach of pure PDE techniques.

- Interaction of N-lumps with a line soliton for the focusing DS II (Fokas-Pelinovsky-C. Sulem 2001).

- The results obtained by IST techniques on the (integrable) DS II system lead to conjecture that pure dispersion governs the dynamics in the defocusing case (and probably in the focusing, non-integrable case). This is also suggested by numerical simulations (C.Klein, JCS 2014).



DS II, $\beta = 0$, $\psi_0(x, y) = \exp(-x^2 - y^2)$, fit of the sup norm with $1/t$.



DS II, $\beta = 0.9$, $\epsilon = 0.1$, $\psi_0(x, y) = \exp(-x^2 - y^2)$, at different times

DS I type systems

- The DS I type systems are quite different from the other DS systems. Actually, solving the hyperbolic equation for ϕ (with suitable conditions at infinity) yields a *loss of one derivative* in the nonlinear term.
- Even the rigorous conservation of the Hamiltonian leads to serious problems.

The elliptic-hyperbolic DS system can be written after scaling as

$$\begin{cases} i\partial_t\psi + \Delta\psi = \chi|\psi|^2\psi + b\psi\phi_x \\ \phi_{xx} - c^2\phi_{yy} = \dfrac{\partial}{\partial x}|\psi|^2. \end{cases}$$

- The integrable DS I sytem corresponds to $\chi + \frac{b}{2} = 0$.

Solvability of the equation for ϕ.

Let $c > 0$. Consider the equation

$$\frac{\partial^2 \phi}{\partial x^2} - c^2 \frac{\partial^2 \phi}{\partial y^2} = f \quad \text{in } \mathbb{R}^2 \tag{42}$$

with the boundary condition

$$\lim_{\xi \to +\infty} \phi(x, y) = \lim_{\eta \to +\infty} \phi(x, y) = 0 \tag{43}$$

where $\xi = cx - y$ and $\eta = cx + y$.
Let $K = K_c$ the kernel

$$K(x, y; x_1, y_1) = \frac{1}{2} H(c(x_1 - x) + y - y_1) H(c(x_1 - x) + y_1 - y)$$

where H is the usual Heaviside function.

Lemma

(Ghidaglia-JCS 1990). Then, for every $f \in L^1(\mathbb{R}^2)$*, the function* $\phi = \mathcal{K}(f)$ *defined by*

$$\phi(x,y) = \int_{\mathbb{R}^2} K(x,y;x_1,y_1) f(x_1,y_1)\, dx_1 dy_1 \tag{44}$$

is continuous on $\mathbb{R}^2$ *and satisfies* (42) *in the sense of distributions. Moreover,* $\phi \in L^\infty(\mathbb{R}^2), (\partial\phi/\partial x)^2 - c^2(\partial\phi/\partial y)^2 \in L^1(\mathbb{R}^2)$ *and we have the following estimates*

$$\sup_{(x,y)\in\mathbb{R}^2} |\phi(x,y)| \le \int_{\mathbb{R}^2} |f| dxdy \tag{45}$$

$$\int_{\mathbb{R}^2} \left| \left(\frac{\partial\phi}{\partial x}\right)^2 - c^2 \left(\frac{\partial\phi}{\partial y}\right)^2 \right| dxdy \le \frac{1}{2c} \left(\int_{\mathbb{R}^2} |f| dxdy \right)^2 . \tag{46}$$

Remark

1. *No condition is required as* ξ *or* η *tends to* $-\infty$.
2. *In general,* $\nabla\phi \notin L^2(\mathbb{R}^2)$ *even if* $f \in C_0^\infty(\mathbb{R}^2)$, *but Lemma 8 allows to solve the* ϕ *equation as soon as* $\psi \in H^1(\mathbb{R}^2)$ *for instance.*

The DS I type system possesses the formal Hamiltonian

$$E(t) = \int_{\mathbb{R}^2} \left[|\nabla \psi|^2 + \frac{\chi}{2}|\psi|^4 + \frac{b}{2}(\phi_x^2 - c^2\phi_y^2) \right] dxdy$$

Lemma 8 allows to prove that this Hamiltonian makes sense in an H^1 setting for ψ (see (GS). Proving its conservation on the time interval of the solution is an open problem as far as we know (this would lead to global existence of a weak solution).

DS I type by PDE methods

- The first result is due to Linares-Ponce (1993) and the best known results are due to Hayashi-Hirata (1996) and Hayashi (1997).

After rotation, one can write the DS-I type systems as

$$\begin{cases} i\partial_t\psi + \Delta\psi = i(c_1 + \frac{c_2}{2})|\psi|^2\psi - \frac{c_2}{4}\left(\int_x^\infty \partial_y|\psi|^2dx' + \int_y^\infty \partial_x|\psi|^2dy'\right)\psi \\ + \frac{c_2}{\sqrt{2}}\left((\partial_x\phi_1) + \partial_y\phi_2)\right)\psi, \end{cases}$$

where $c_1, c_2 \in \mathbb{R}$ and ϕ satisfies the radiation conditions

$$\lim_{y\to\infty}\phi(x,y,t) = \phi_1(x,t), \quad \lim_{x\to\infty}\phi(x,y,t) = \phi_2(y,t).$$

Theorem

(Hayashi 1997)
Assume $\psi_0 \in H^2(\mathbb{R}^2)$, $\phi_1 \in C(\mathbb{R}; H^2_x)$ *and* $\phi_2 \in C(\mathbb{R}; H^2_y)$. *Then there exist* $T > 0$ *and a unique solution* $\psi \in C([0, T]; H^1) \cap L^\infty(0, T; H^2)$ *with initial data* ψ_0.

- The proof uses in a crucial way the smoothing properties of the Schrödinger group.

Global existence and scattering of small solutions (Hayashi-Hirata 1996).

$$H^{m,l} = \{f \in L^2(\mathbb{R}^2); |(1-\partial_x^2-\partial_y^2)^{m/2}(1+x^2+y^2)^{l/2} f|_{L^2} < \infty\}.$$

Theorem

Let $\psi_0 \in H^{3,0} \cap H^{0,3}$, $\partial_x^{j+1}\phi_1 \in C(\mathbb{R}; L_x^\infty)$, $\partial_y^{j+1}\phi_2 \in C(\mathbb{R}; L_y^\infty)$, $(0 \le j \le 3)$, "small enough". Then

- *There exists a solution $\psi \in L^\infty_{loc}(\mathbb{R}; H^{3,0} \cap H^{0,3}) \cap C(\mathbb{R}; H^{2,0} \cap H^{0,2})$.*
- *Moreover*

$$||\psi(\cdot,t)||_{L^\infty} \le C(1+|t|)^{-1}(||\psi||_{H^{3,0}} + ||\psi||_{H^{0,3}}).$$

There exist $u^\pm$ such that

$$||\psi(t) - U(t)u^\pm||_{H^{2,0}} \to 0, \quad as \ \to \pm\infty.$$

where $U(t) = e^{it(\partial_x^2+\partial_y^2)}$.

DS I by IST. Comparison with elliptic-hyperbolic DS

- Existence of coherent structures (dromions) for DS I with non-trivial boundary conditions (Boiti-Leon-Martina-Pempinelli 1988, Fokas-Santini 1990).
- Probably not physically relevant.
- Stability of the dromion (Kiselev (2000).
- I don't know of similar results in the non-integrable case, or any stability analysis by PDE techniques.

- Global existence and uniqueness of a solution $\psi \in C(\mathbb{R}; \mathcal{S}(\mathbb{R}^2))$ of DS I for data $\psi_0 \in \mathcal{S}(\mathbb{R}^2),\ \phi_1, \phi_2 \in C(\mathbb{R}; \mathcal{S}(\mathbb{R}))$ (Fokas-Sung 1992). The solutions with trivial boundary conditions $\phi_1 = \phi_2 = 0$ disperse as $1/t$ (Kiselev 1998).
- Numerical simulations for elliptic-hyperbolic DS systems, including DS I (Besse-Bruneau 1998) confirm the dispersion of solutions of DS I with trivial boundary conditions and suggest that the dromion is not stable with respect to the coefficients, that is it does not persist in the non-integrable case (need more work...).

Numerical simulations of elliptic-hyperbolic DS (Besse-Bruneau 1998)

They consider the version

$$\begin{cases} i\partial_t\psi + \Delta\psi = -|\psi|^2\psi + b\psi\phi_{\xi+\eta} \\ \phi_{\xi\eta} = \frac{\sigma}{4}(|\psi|^2_\xi + |\psi|^2_\eta), \quad \xi = cx - y,\ \eta = cx + y. \end{cases}$$

- Observe that when $b = 0$ (focusing cubic NLS) there is a blow-up in finite time, say from a gaussian initial data $\psi_0(x, y) = 4\exp(-x^2 - y^2)$.
- A finite time blow-up seems to occur when $\sigma > 0$ and $b > 0$. When $\sigma < 0$, a stabilization seems to occur when $b < 0$.

The elliptic-elliptic case

$$\begin{cases} i\partial_t\psi + \Delta\psi = \chi|\psi|^2\psi + b\psi\phi_x \\ \Delta\phi = \partial_x|\psi|^2. \end{cases}$$

- Inverting the equation for ϕ reduces to a cubic NLS with the extra nonlocal (order zero) term $-b\psi(-\Delta)^{-1}\partial_x^2|\psi|^2$.
- Energy :

$$E(\psi) = \int_{\mathbb{R}^2} \left(|\nabla\psi|^2 + \frac{1}{2}(\chi|\psi|^4 + b(\phi_x^2 + \phi_y^2)) \right) dxdy.$$

- Local well-posedness on $[0, T^*]$ for initial data ψ_0 in L^2, H^1, and $\Sigma = \{f \in H^1(\mathbb{R}^2); (x^2+y^2)^{1/2} f \in L^2(\mathbb{R}^2)\}$. (Ghidaglia-JCS 1990).

A blow-up result (Ghidaglia-JCS 1990)

Theorem

Let Σ_- the set of $u \in \Sigma$ such that $E(u) < 0$. Then

1. *The set Σ_- is not empty if and only if $\chi < \max(-b, 0)$.*
2. *For $\chi \geq \max(-b, 0)$, the local solution is global : $T^* = +\infty$.*
3. *For $\chi < \max(-b, 0)$, and for every $\psi_0 \in \Sigma_-$, the maximal solution on $[0, T^*)$ satisfies $T^* = +\infty$.*

- A refined analysis of the blow-up *à la Merle-Raphël (2005)* is missing.
- Existence and stability properties of ground states solutions : R. Cipolatti (1992,1993), Sulem-Sulem-Wang (1993), M. Ohta (1995).
- Numerical simulations of the blow-up : Papanicolaou-Sulem-Sulem-Wang (1994).

Some conclusions

- In the KdV and DS II cases, IST provides results out of reach of pure PDE techniques and might give hints on the dynamics of "close" non-integrable equations.

  Note however that some properties of DS I (existence of dromions), or of the focusing DS II (existence of lumps, blow-up in finite time) do not seem to persist in the corresponding non-integrable cases.
- For BO and KP equations, PDE techniques provide nice GWP results, without information on the dynamics.
- The DS I type systems need more numerical simulations.
- An important issue (not seriously discussed here) is to decide whether or not the various phenomena observed in the asymptotic models are relevant to the original system (eg the water wave system). Actually they might occur after the model has ceased to be a relevant approximation.

- In the modulation regime for instance, full dispersion models are probably more relevant (valid for a larger band of frequencies), see D. Lannes (2013) and C. Obrecht 2014 for Benney-Roskes and DS type full dispersion systems.None of them is integrable...

The full dispersion DS system

$$\begin{cases} \partial_\tau \psi + \dfrac{i}{\varepsilon^2}[\omega(\mathbf{k}+\varepsilon D') - \omega - \varepsilon\omega' D'_x]\psi \\ + i(\beta\partial'_x\phi + 2\dfrac{|\mathbf{k}|^4}{\omega}(1-\tilde{\alpha})|\psi|^2)\psi = 0 \\ \left(\dfrac{|D'|\tanh(\varepsilon\sqrt{\mu}|D'|)}{\varepsilon} + \omega'^2\partial'^2_x\right)\phi = 2\omega\beta\partial'_x|\psi|^2. \end{cases} \tag{47}$$

Here the time and space derivatives (indicated by a $'$) are taken with respect to the slow time and space variables $t' = \varepsilon t$ and $X' = \varepsilon X$ and D' denotes the operator $\frac{1}{i}\nabla'$. We have used furthermore used the following notations

$$\begin{cases} \mathbf{k} = |\mathbf{k}|\mathbf{e}_x, \quad \omega(\mathbf{k}) = \tilde{\omega}(|\mathbf{k}|), \quad \text{with} \quad \tilde{\omega}(r) = (r\tanh(\sqrt{\mu}r))^{1/2}, \\ \omega = \tilde{\omega}(|\mathbf{k}|), \quad \omega' = \tilde{\omega}'(|\mathbf{k}|), \quad \omega'' = \tilde{\omega}''(|\mathbf{k}|) \\ \sigma = \tanh(\sqrt{\mu}|\mathbf{k}|), \quad \alpha = -\dfrac{9}{8\sigma^2}(1-\sigma^2)^2, \quad \tilde{\alpha} = \alpha + \dfrac{1}{4}(1-\sigma^2)^2, \\ \beta = |\mathbf{k}|\left(1 + (1-\sigma^2)\dfrac{\omega'|\mathbf{k}|}{2\omega}\right). \end{cases} \tag{48}$$