Toupin, R. A. 1952 A variational principle for mesh-type analysis of a mechanical sys m. *J. appl. Mech.* **19** (2), 151–152.

Whittaker, E. 1917 *Analytical dynamics*. Cambridge University Press.

Xing, J. T. 1984 Some theoretical and computational aspects of finite element method and substructure–subdomain technique for dynamic analysis of the coupled fluid–solid interaction problems – variational principles for elastodynamics and linear theory of micropolar elasticity with their applications to dynamic analysis. Ph.D. dissertation, Department of Mechanical Engineering Mechanics, Qinghua University, Beijing, People's Republic of China. (In Chinese.)

Xing, J. T. 1986*a* Finite element-substructure method for dynamic analysis of coupled fluid–solid interaction problems. In *Proc. Int. Conf. Comp. Mech.*, vol. IX, pp. 117–121. Tokyo: Springer-Verlag.

Xing, J. T. 1986*b* A study on finite element method analysis of coupled fluid–solid interaction problems. *Acta Mech. Solida Sinica* **4**, 229–237. (In Chinese).

Xing, J. T. & Zheng, Z. C. 1986 Some general theorems and generalized and piece-generalized variational principles for elastodynamics. In *Proc. China-American Workshop on F. E. M.*, pp. 253–267. Beijing: BIAA Press.

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Exact blow-up solutions to the Cauchy problen the Davey–Stewartson systems

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We present exact blow-up solutions to the Cauchy problem for the] Stewartson systems. It is shown that for any prescribed blow-up time thei exact solution whose mass density converges to the Dirac measure as time goe blow-up time and that the solution extends beyond the blow-up time and b like the free solution as time tends to infinity.

1. Introduction

The time evolution of two-dimensional surfaces of water waves has been descri Davey & Stewartson (1974) and Djordejevic & Redekopp (1977) und assumption that the water waves are subject to (1) weakly nonlinear modul (2) slowly varying modulations, (3) propagation along nearly x-direction, an balance of these three effects. In suitably rescaled coordinates, the I Stewartson (DS) systems are the following:

$$\mathrm{i}\,\partial_t u + \sigma\,\partial_x^2 u + \partial_y^2 u = \lambda |u|^2 u + \mu\,\partial_x \varphi \cdot u,$$
$$\partial_x^2 \varphi + m\,\partial_y^2 \varphi = \partial_x |u|^2.$$

Here u is a complex-valued function of space and time variables (x, y, t) anc real-valued function of (x, y, t). The four parameters $(\sigma, \lambda, \mu, m)$ are real and have been normalized as $|\sigma| = |\lambda| = 1$. The functions u and φ are related amplitude and the mean velocity potential of the water wave, respectively.

For some special choices of parameters DS become integrable systems and s transform techniques can be applied (see Anker & Freeman (1978), Fokas & ! (1988) and their references). In the full general case except the hyperbolic–hyp case, i.e. $\sigma, m < 0$, Ghidaglia & Saut (1990) studied the Cauchy problem for functional analytic methods similar to those of the nonlinear Schrödinger eqı i.e. $\sigma > 0, \mu = 0$ (see Ginibre & Velo 1979; Brezis & Gallouet 1980; Kato Cazenave 1989) and proved the solvability in the Lebesgue space $L^2 = L^2(\mathbb{R}^2)$ ε Sobolev space $H^1 = H^1(\mathbb{R}^2)$. In the elliptic–hyperbolic case, i.e. $\sigma > 0$, Tsutsumi (1991) obtained $L^p(\mathbb{R}^2)$-decay estimates of the solutions for any $2 < p < \infty$.

The purpose of this paper is to study the blow-up or self-focusing propeı solutions of DS. In the elliptic–elliptic case, i.e. $\sigma, m > 0$, Ghidaglia & Saut studied the blow-up problem. In this case the blow-up phenomena is e analogous to that of the nonlinear Schrödinger equations (see Glassey Tsutsumi 1984; Weinstein 1986; Nawa & Tsutsumi 1989; Merle & Tsutsumi

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In this paper, following Nawa & Tsutsumi (1989) and Weinstein (1986), e present exact blow-up solutions of DS in the hyperbolic–elliptic case, i.e. $\sigma < 0, m > 0$. Moreover, we describe discrepancies between the nonlinear Schrödinger equation and DS in the hyperbolic–elliptic case.

2. Invariance of the Davey–Stewartson systems

We start with some basic character of DS. DS are invariant under the following transformations: (1) Time translations $t \to t+t_0$. (2) Space translations $(x, y) \to (x+x_0, y+y_0)$. (3) Phase transformations $(u, \varphi) \to (e^{i\theta}u, \varphi), \theta \in \mathbb{R}$. The associated conserved quantities are given respectively by (see Ghidaglia & Saut 1990):

$$E(u) = \sigma\|\partial_x u\|_2^2 + \|\partial_y u\|_2^2 + \tfrac{1}{2}(\lambda\|u\|_4^4 + \mu\|\partial_x\varphi\|_2^2 + \mu m\|\partial_y\varphi\|_2^2), \tag{3}$$

$$P(u) = \operatorname{Im}(u, \nabla u), \tag{4}$$

$$M(u) = \|u\|_2^2. \tag{5}$$

Here $\|\cdot\|_p$ denotes the $L^p(\mathbb{R}^2)$-norm and $(\cdot,\cdot)$ denotes the L^2-scalar product. E, P, and M are the energy, momentum, and mass, respectively. DS have another special invariance $(u, \varphi) \to (Cu, \tilde{C}\varphi)$, where

$$(Cu)(x, y, t) = (a+bt)^{-1}\exp(ib(4(a+bt))^{-1}(\sigma x^2+y^2))\,u(X, Y, T), \tag{6}$$

$$(\tilde{C}\varphi)(x, y, t) = (a+bt)^{-1}\varphi(X, Y, T), \tag{7}$$

$$X = (a+bt)^{-1}x, \quad Y = (a+bt)^{-1}y, \quad T = (a+bt)^{-1}(c+dt),$$

and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{R}),$$

i.e. $a, b, c, d \in \mathbb{R}$ and $ad-bd = 1$. This may be the first time that the transformations (6) and (7) are noticed, although (6) has been known for the nonlinear Schrödinger equation (see Ginibre & Velo 1982; Weinstein 1986; Nawa & Tsutsumi 1989; Cazenave & Weissler 1990, 1992). The associated conserved quantity is

$$\sigma\|(x+2i\sigma t\,\partial_x)u\|_2^2 + \|(y+2it\,\partial_y)u\|_2^2 + 2t^2(\lambda\|u\|_4^4 + \mu\|\partial_x\varphi\|_2^2 + \mu m\|\partial_y\varphi\|_2^2). \tag{8}$$

The corresponding conserved quantity is well known for the nonlinear Schrödinger equation.

3. Exact blow-up solutions

We look for exact blow-up solutions by using the transformations $(C, \tilde{C})$. By the analogy with the well-known lump solution for the KdV equation, we first seek special solutions of the simple form

$$u(x, y, t) = 1/f(x, y), \quad \varphi(x, y, t) = \gamma\,\partial_x f(x, y)/f(x, y) = \gamma\,\partial_x(\ln f(x, y)),$$

where $f(x, y) = 1+\alpha x^2+\beta y^2$. Substituting φ into the left-hand side of (2), we have

$$\partial_x^2\varphi + m\,\partial_y^2\varphi = 2\gamma\,\partial_x(((\alpha+\beta m)+\alpha(\beta m-\alpha)x^2+\beta(\alpha-\beta m)y^2)/f^2),$$

so that we obtain $\alpha = \beta m$ and $\gamma = \frac{1}{4}\alpha$. Similarly, we have

$$\begin{aligned} &i\partial_t u + \sigma\partial_x^2 u + \partial_y^2 u - \lambda|u|^2 u - \mu\,\partial_x\varphi\cdot u \\ &= -((4\alpha\sigma+4\beta+2\lambda+\mu) - \alpha(12\alpha\sigma-4\beta+\mu)x^2 + \beta(4\alpha\sigma-12\beta+\mu)y^2)/(2f^3), \end{aligned}$$

which implies $m\sigma = -1$ and $\mu = -2\lambda = 16\beta$. Therefore,

$$v_\alpha(x, y) = (1+\alpha(x^2-\sigma y^2))^{-1},$$

$$\psi_\alpha(x, y) = \tfrac{1}{4}x(1+\alpha(x^2-\sigma y^2))^{-1}$$

are solutions of DS provided that $m\sigma = -1$ and $\mu = -2\lambda = 16\alpha/m$. When and $\alpha > 0$, we have $v_\alpha \in L^2$ and

$$\|v_\alpha\|_2 = (\pi/\alpha)^{\frac{1}{2}}.$$

Now let $u_\alpha = Cv_\alpha$, $\varphi_\alpha = \tilde{C}\psi_\alpha$ with $\alpha > 0$, $ab \neq 0$. Let $m = 1$, $\sigma = -1$, $\mu =$ $16\alpha/m$. Then DS is hyperbolic–elliptic type and $(u_\alpha, \varphi_\alpha)$ solves DS. Th condition becomes

$$u_\alpha(x, y, 0) = a^{-1}\exp(-ib(4a)^{-1}(x^2-y^2))\,v_\alpha(x, y).$$

Moreover, for any $t \in \mathbb{R}$ with $a+bt \neq 0$

$$\|u_\alpha(t)\|_2 = \|v_\alpha\|^2 = (\pi/\alpha)^{\frac{1}{2}},$$

while by a direct calculation,

$$(x^2+y^2)^{\frac{1}{2}}u_\alpha(t), \quad \nabla u_\alpha(t) \notin L^2$$

for any $t \in \mathbb{R}$. Therefore our solution u_α is an L^2-solution, but not an H^1-sol any case. On the other hand,

$$(x^2+y^2)^{s/2}u_\alpha(t), (-\Delta)^{s/2}u_\alpha(t) \in L^2$$

for any $s, t \in \mathbb{R}$ with $0 < s < 1$ and $a+bt \neq 0$. The last assertion follows by u Besov semi-norms defined in terms of the modulus of continuity and em theorems of the homogeneous Sobolev spaces in a way similar to the proof of 2.4 in Hayashi & Ozawa (1988).

We now describe the formation of blow-up of the solution $u_\alpha(t)$. For simpl assume $ab < 0$ and put $T = -a/b$. The mass density takes the form

$$|u_\alpha(x, y, t)|^2 = \epsilon^{-2}(1+\alpha\epsilon^{-2}(x^2+y^2))^{-2} = \epsilon^{-2}|v_\alpha(\epsilon^{-1}x, \epsilon^{-1}y)|^2$$

with $\epsilon = a+bt = -b(T-t)$. By (11), we find

$$|u_\alpha(t)|^2 \to (\pi/\alpha)\,\delta \quad \text{in} \quad \mathcal{S}' \quad \text{as} \quad t \to T,$$

where δ denotes the Dirac measure at the origin. Moreover, for any s with 0 we have by letting $t \to T$,

$$\|(x^2+y^2)^{s/2}u_\alpha(t)\|_2 = |b|^s|T-t|^s\|(x^2+y^2)^{s/2}v_\alpha\|_2 \to 0,$$

$$\begin{aligned} \|(-\Delta)^{s/2}u_\alpha(t)\|_2 &\geq C\|(x^2+y^2)^{-s/2}u_\alpha(t)\|_2 \\ &= C|b|^{-s}|T-t|^{-s}\|(x^2+y^2)^{-s/2}v_\alpha\|_2 \to \infty, \end{aligned}$$

where we have used the Hardy type inequality (see Herbst 1977).

Therefore the blow-up process of the solution $u_\alpha(t)$ is summarized as follo start with $u_\alpha(0)$ given by (12). The initial state $u_\alpha(0)$ looks like a lump aroı origin and has an algebraic decay in space, or equivalently, in two hoı directions. Due to the phase factor, $u_\alpha(0)$ is not radial. By Ghidaglia uniqueness result in L^2, the unique solutions of DS are given by $u_\alpha = Cv_\alpha, \varphi_\alpha$ As time increases, the solution $u_\alpha(t)$ is localized at the origin and the mass $|u_\alpha(t)|^2$ takes its maximum value $\epsilon^{-2} = b^{-2}(T-t)^{-2}$ at the origin. As time goe blow-up time T, $|u_\alpha(t)|^2$ tends to the Dirac measure with total mass $\|u_\alpha(t)\|_2^2$ conserved. Every regularity breaks down at the blow-up time as described in (16). The solution $u_\alpha(t)$, however, exists after the blow-up time and gains the original regularity. As time goes to infinity, the mass density is dispersed from the origin and decays like t^{-2}. If we reverse the time direction, we see that for negative times the solution $u_\alpha(t)$ is stable in the sense that the blow-up phenomenon does not occur.

When $ab > 0$, we have another blow-up solution with negative blow-up time which is stable for positive times.

The blow-up process is a natural consequence of the nonlinearity since the linear partial differential operator $\mathrm{i}\,\partial_t - \partial_x^2 + \partial_y^2$, or rather the associated propagator $U(t) = \exp(it(-\partial_x^2+\partial_y^2))$ has the dispersive and smoothing effects. The effect of the nonlinearity, however, is dominant up to the blow-up time and loses its influence for large times because of the dispersive effect of the linear term. To be more specific, the solution $u_\alpha(t)$ behaves like the free solution in the sense that there is a unique $f \in L^2$ satisfying

$$\|u_\alpha(t) - U(t)f\|_2 \to 0 \quad \text{as} \quad t \to \pm\infty. \tag{17}$$

Indeed, f is given explicitly by $f = U(a/b)\,g$ with

$$g(x,y) = \mathrm{i}(2\alpha b)^{-1} K_0((4\alpha)^{-\frac{1}{2}}(x^2+y^2)^{\frac{1}{2}}),$$

where K_0 is the modified Bessel function of the second kind of order zero. The proof uses the following three facts:

(1) $U(t)$ has the representation $U(t)\,\psi = S(t)\,D(t)\,\mathscr{F}S(t)\,\psi_\sigma$, where

$$S(t) = \exp(\mathrm{i}(4t)^{-1}(-x^2+y^2)), \quad (D(t)\,\psi)(x) = (2\mathrm{i}t)^{-1}\psi((2t)^{-1}x, (2t)^{-1}y),$$

$$(\mathscr{F}\psi)(\xi,\eta) = (2\pi)^{-1}\int_{\mathbb{R}^2} \exp(-\mathrm{i}x\xi - \mathrm{i}y\eta)\,\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y, \quad \psi_\sigma(x,y) = \psi(-x,y).$$

(2) $S(t) \to \mathbb{1}$ strongly on L^2 as $t \to \pm\infty$.

(3) $(\mathscr{F}v_\alpha)(\xi,\eta) = \alpha^{-1}K_0(\alpha^{-\frac{1}{2}}(\xi^2+\eta^2)^{\frac{1}{2}})$.

We finally remark that in view of (13) the total mass of the initial data $u_\alpha(0)$ may be chosen arbitrarily small or large by changing α. Consequently, there are blow-up solutions with total mass arbitrarily small. In addition to the property (17), this is a sharp contrast to the case of the nonlinear Schrödinger equation. These new features reflect the hyperbolicity of the operator $-\partial_x^2 + \partial_y^2$.

References

Anker, D. & Freeman, N. C. 1978 On the soliton solutions of the Davey–Stewartson equation for long waves. *Proc. R. Soc. Lond.* A **360**, 529–540.

Brezis, H. & Gallouet, T. 1980 Nonlinear Schrödinger evolution equations. *Nonlinear Analysis TMA* **4**, 677–681.

Cazenave, T. 1989 *Nonlinear Schrödinger equations*. Textos de Metódos Matemáticos, vol. 22. Rio de Janeiro: Institudo de Matemática.

Cazenave, T. & Weissler, F. B. 1990 Rapidly decaying solutions of the nonlinear Schrödinger equation. Preprint, ENS Cachan.

Cazenave, T. & Weissler, F. B. 1992 The structure of the solutions to pseudo-conformally invariant nonlinear Schrödinger equation. *Proc. R. Soc. Edinb.* (In the press.)

Davey, A. & Stewartson, K. 1974 On three-dimensional packets of surface waves. *Proc. R. Soc. Lond.* A **338**, 101–110.

Djordjevic, V. D. & Redekopp, L. G. 1977 On two-dimensional packets of capillary-gravity waves. *J. Fluid Mech.* **79**, 703–714.

Fokas, A. S. & Santini, P. M. 1988 Recursion operators and bi-Hamiltonian structu multidimensions I, II. *Communs math. Phys.* **115**, 375–419; **116**, 449–474.

Ghidaglia, J. M. & Saut, J. C. 1990 On the initial value problem for the Davey–Stew systems. *Nonlinearity* **3**, 475–506.

Ginibre, J. & Velo, G. 1979 On a class of nonlinear Schrödinger equations. I. The Cauchy pr general case. *J. Funct. Analysis* **32**, 1–32.

Ginibre, J. & Velo, G. 1979 On a class of nonlinear Schrödinger equations. II. Scattering t general case. *J. Funct. Analysis* **32**, 33–71.

Ginibre, J. & Velo, G. 1982 Sur une équation de Schrödinger nonlinéaire avec interactio locale. In *Nonlinear partial differential equations and their applications*. College de France Se vol. II (ed. H. Brezis & J. L. Lions), pp. 115–199. Research Notes in Mathematics, v London: Pitman.

Glassey, R. T. 1977 On the blowing up of solutions to the Cauchy problem for non Schrödinger equations. *J. math. Phys.* **18**, 1794–1797.

Hayashi, N. & Ozawa, T. 1988 Scattering theory in the weighted $L^2(\mathbb{R}^n)$ spaces for Schrödinger equations. *Ann. Inst. Henri Poincaré, Phys. théor.* **48**, 17–37.

Herbst, I. W. 1977 Spectral theory of the operator $(p^2+m_2)^{\frac{1}{2}} - Ze^2/r$. *Communs math. Phy* 285–294.

Kato, T. 1987 On nonlinear Schrödinger equations. *Ann. Inst. Henri Poincaré, Phys. théo* 113–129.

Merle, F. & Tsutsumi, Y. 1990 L^2 concentration of blow up solutions for the nonlinear Schrö equation with critical power nonlinearity. *J. diffl Equats* **84**, 205–214.

Nawa, H. & Tsutsumi, M. 1989 On blow-up for the pseudo-conformally invariant non Schrödinger equation. *Funkcialaj Ekvac.* **32**, 417–428.

Tsutsumi, M. 1984 Nonexistence of global solutions to the Cauchy problem for the da nonlinear Schrödinger equation. *SIAM J. math. Analysis* **15**, 357–366.

Tsutsumi, M. 1991 Decay of weak solutions of the Davey–Stewartson systems. Preprint.

Weinstein, M. 1986 On the structure and formation of singularities in solutions to nonl dispersive evolution equations. *Commun. partial diffl Equats* **11**, 545–565.

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