OPEN PROBLEMS IN DISPERSIVE PDE

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ABSTRACT. This is a list of open problems compiled from discussion on Saturday, May 17. I've selected out those involving the ISM

(1) (suggested by Peter Miller) Prove that, in the semiclassical limit, the reflection coefficient for the defocussing DS II equation is $\mathcal{O}(\varepsilon)$ as $\varepsilon \downarrow 0$. Recall that the semiclassical DS II equation is

$$\begin{aligned} i\varepsilon_t + 2\varepsilon^2 \left(\overline{\partial}^2 + \partial^2\right) q + (g + \overline{g})q &= 0 \\ \overline{\partial} g + \partial\left(|q|^2\right) &= 0 \end{aligned}$$

(2) (Suggested by Kari Astala) Could someone provide an example of a nonlocal Riemann-Hilbert problem, and carry out the subsequent analysis?

Remark One reference is the paper of X. Zhou (*Comm. Math. Phys.* **128** (1990), 551) on the KP I equation and its nonlocal Riemann-Hilbert problem.

(3) (suggested by Andreas Stahel) Prove global well-posedness for the Novikov-Veselov equation, and also find a precise connection to conservation laws via the inverse scattering transform. Even a proof for small data would be interesting.

Remark Some progress on GWP has been made by Perry and Music-Perry, but only for spectrally determined classes of initial data.

(4) (Suggested by J.-C. Saut) Show that (or disprove) the "lump" solution of KP I is a minimizer of the functional

$$E(u) = \int_{\mathbb{R}^2} \left[\frac{1}{2}u_x^2 - \frac{u^3}{6} + \frac{1}{2}(\partial_x^{-1}\partial_y u)^2\right] dxdy$$

over u with fixed L^2 norm

(5) (Suggested by Jim Colliander) Prove long-time asytmptotics and stability for N-soliton (N finite) wave train emerging from some general class of initial data for the focusing NLS equation. Even for analytic data, this would be appreciated by PDE community.

Remark This is a natural application of "Riemann-Hilbert" technology. One should bear in mind Deift-Zhou's example of Schwarz class initial data for cubic focussing NLS with infinitely many singularities. Can one show that, for analytic initial data, there can be at most finitely many solitons?

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(6) (Suggested by Paolo Santini) Extend Peter Perry's work on the DS II equation to the following more general nonlinear term: replace $\overline{\partial}^{-1}\partial\left(|u|^2\right)$ by

$$\overline{\partial}^{-1}\partial\left(|u|^2\right) + g(z,t)$$

for a given function of z and t, meromorphic in z, with decay properties to be specified.

One is looking for the analogue of the dromion solutions of DS I. The expectation is new solutions which are not exponentially localized.

(7) (Suggested by Paolo Santini) Find the (temporal) scattering map for the DS-II equation. Is it trivial or not?

Remark: Let's recall the definition of the (temporal) scattering maps. For an evolution equation on a Banach space X of the form

$$iu_t = Lu + NL(u)$$

where L is a linear, self-adjoint operator and $NL(\,\cdot\,)$ is the nonlinear term, the corresponding linear equation is

$$iv_t = Lu$$

Suppose that u_0 is Cauchy data for the nonlinear equation, and let $u(t)$ be the solution of the nonlinear problem. Let $S(t)$ be the solution operator for the *linear* problem. Suppose that for each $u_0 \in X$, there are unique vectors $v_\pm$ in X with

$$\lim_{t\to-\infty} \|u(t) - S(t)v_-\| = 0$$
$$\lim_{t\to+\infty} \|u(t) - S(t)v_+\| = 0$$

Then the map $\mathcal{S} : v_- \to v_+$ is called the (temporal) *scattering map.*

Similar questions can be asked for other 2+1 equations.

(8) (suggested by J.-C. Saut) Prove a scattering result for KP II similar to Perry's result for the DS II equation.

Remark Inverse scattering for KP II is defined by a $\overline{\partial}$-problem very similar to that studied by Perry. The crucial difference is that the phase function is cubic and therefore has degenerate critical points. In fact, the problem takes the form:

$$\overline{\partial}_k M(z,k,t) = F(k)e_k(x,y)e^{-4it(k^3+\overline{k}^3)}M(z,k,t)$$
$$\lim_{|k|\to\infty} M(z,k,t) = 1$$

$$u(x,y,t) = \partial_x\left(\int F(k)M(x,y,-\overline{k})e_k(x,y)e^{-4it(k^3+\overline{k}^3)}\,dA(k)\right)$$

where

$$e_k(x,y) = \exp\left[-i(k+\overline{k})x + (k^2-\overline{k}^2)y\right]$$

Thus the "stationary phase" analysis is a bit more subtle. The phase function is

$$S(x,y,k,t) = -\frac{-(k+\overline{k})x + (k^2-\overline{k}^2)y}{t} - 4t(k^3+\overline{k}^3)$$

and the stationary condition is

$$-\frac{x}{t} + 2k\frac{y}{t} - 8tk^2 = 0$$

which has two roots which coalesce at 0 as $t \to \infty$. Very similar problems arise in the study of the Novikov-Veselov equation.

(9) (suggested by J.-C. Saut) Prove that the localized lump solutions for the focussing DS II equation does not persist in the non-integrable case.

(10) (suggested by J.-C. Saut) Prove that blow-up for focussing DS II may occur for initial data different from the Ozawa solutions–say for a Gaussian.

Remark. It might help to have a spectral characterization of initial data which lead to blowing up solutions. One conjecture is that singular curves for the scatttering data $r(k)$ lead to blow-up in finite time. As a first step, it would be very nice to compute the scattering transform for Ozawa's potential and see how it evolves in time.

More generally, it would be helpful to know what kinds of singularities occur. Here the determinant might be very helpful.

Recall the set-up for solving DS II by inverse scattering. Given the scattering data $r(k)$, one solves the $\overline{\partial}$-problem

$$\begin{aligned}\overline{\partial}_k \mu_1(z,k,t) &= e^{-itS}\overline{r(k)\mu_2(z,k,t)}\\ \overline{\partial}_k \mu_2(z,k,t) &= -e^{-itS}\overline{r(k)}\overline{\mu_1}(z,k,t)\end{aligned}$$

$$\lim_{|k|\to\infty}(\mu_1(z,k,t),\mu_2(z,k,t)) = (1,0)$$

where

$$S(z,k,t) = -\frac{kz + \overline{k}\overline{z}}{t} + 2i(k^2 + \overline{k}^2)$$

(11) (suggested by J.-C. Saut) Can we solve KP I, KP II, Benjamin Ono by IST for arbitrary initial data (not just "small" initial data)?

Remark KP I and BO have solitons so the problem will be controlling the spectrum, and in particular the exceptional points. It would be very helpful, for example, to know that there are only finitely many singularities for some class of potentials