The KPII on the half-plane

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The problem

We consider the following initial-boundary value problem for the Kadomtsev-Petviashvili II equation:

$$
\begin{aligned}
&q_t + 6qq_x + q_{xxx} + 3\partial_x^{-1} q_{yy} = 0, && (x,y) \in \mathbb{R} \times \mathbb{R}^+,\ t \in (0,T), \\
&q(x,y,0) = q_0(x,y), && (x,y) \in \mathbb{R} \times \overline{\mathbb{R}^+}, \\
&q(x,0,t) = g(x,t) \text{ or } q_y(x,0,t) = h(x,t), && (x,t) \in \mathbb{R} \times [0,T].
\end{aligned}
$$

Recall that the operator ∂_x^{-1} is formally defined by

$$\partial_x^{-1} f(x) = \int_{-\infty}^{x} f(x')dx', \quad x \in \mathbb{R}.$$

Lax pair

For

$$\mu = \mu(x, y, t, k_R, k_I), \quad k_R = \mathrm{Re}(k),\ k_I = \mathrm{Im}(k)$$

it is not hard to check that the KPII is the compatibility condition of the Lax pair

$$\mu_y - \mu_{xx} - 2ik\mu_x = q\mu,$$

$$\mu_t + 4\mu_{xxx} + 12ik\mu_{xx} - 12k^2\mu_x = F\mu,$$

where the operator F is defined by

$$F(x, y, t, k) = -6q\left(\partial_x + ik\right) - 3\left(q_x + \partial_x^{-1} q_y\right).$$

Observe that assuming

$$\mu = 1 + O\left(\frac{1}{k}\right), \quad |k| \to \infty,$$

implies

$$q(x, y, t) = -2i \lim_{|k|\to\infty} \left[k\mu_x(x, y, t, k_R, k_I)\right].$$

- For computational purposes, we treat $\{q_0, g, h\}$ as Schwartz functions. After the formal expressions have been obtained, one should ask what bigger functions spaces can we draw the data from.

- We will seek a solution which decays as $y \to \infty$ for all fixed (x, t) and which also decays as $|x| \to \infty$ for all fixed (y, t).

- We will begin by *assuming* that there exists a solution q with sufficient smoothness and decay.

- We will denote by $\hat{\mu}$ the Fourier transform of $\mu - 1$ in the variable x:

$$\hat{\mu}(l, y, t, k_R, k_I) = \int_{\mathbb{R}} e^{-ilx} \left[\mu(x, y, t, k_R, k_I) - 1\right] dx, \quad l \in \mathbb{R},$$

  with inverse

$$\mu(x, y, t, k_R, k_I) - 1 = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ilx} \hat{\mu}(l, y, t, k_R, k_I) dl, \quad x \in \mathbb{R}.$$

The method

- Our analysis is inspired by the one of Ablowitz, BarYaacov and Fokas used for the KPII Cauchy problem.

- More precisely, it will rely on the formulation of a $\bar{\partial}$ problem.

- Of course, the introduction of the boundary $y = 0$ requires a different approach in order to arrive at this $\bar{\partial}$ problem.

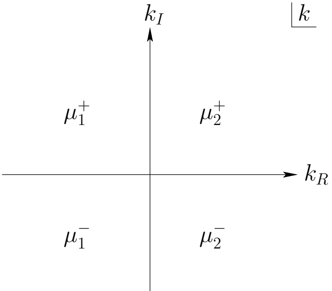
- This approach brings in the novel ideas of Fokas, introduced in 1997, that have since been developed to the so-called unified transform method for initial-boundary value problems, both for linear and for integrable nonlinear PDEs.

Proposition (Direct problem)

Assume that there exists a solution q to the KPII ibvp. Then, there exists a solution μ of the KPII Lax pair which is bounded for all $k \in \mathbb{C}$ and can be represented in the form

$$\mu(x,y,t,k_R,k_I) = \begin{cases} \mu_1^+(x,y,t,k_R,k_I), & k_R < 0,\ k_I > 0, \\ \mu_1^-(x,y,t,k_R,k_I), & k_R < 0,\ k_I < 0, \\ \mu_2^-(x,y,t,k_R,k_I), & k_R > 0,\ k_I < 0, \\ \mu_2^+(x,y,t,k_R,k_I), & k_R > 0,\ k_I > 0, \end{cases}$$

where $\mu_{1,2}^{\pm}$ can be expressed in terms of $\{q(x,y,t), g(x,t), h(x,t)\}$ by



$$\mu_1^{\pm} = 1 + \frac{1}{2\pi}\left(\int_{-\infty}^{0} dl + \int_{-2k_R}^{\infty} dl\right)\int_{-\infty}^{\infty} d\xi \int_0^y d\eta\, e^{-il(\xi-x)+l(l+2k)(\eta-y)} q(\xi,\eta,t)\mu_1^{\pm}$$

$$+ \frac{1}{2\pi}\left(\int_{-\infty}^{0} dl + \int_{-2k_R}^{\infty} dl\right)\int_{-\infty}^{\infty} d\xi \begin{cases} \int_0^t d\tau \\ -\int_t^T d\tau \end{cases} e^{-il(\xi-x)-l(l+2k)y-4il(l^2+3kl+3k^2)(\tau-t)} H(\xi,\tau,k,l)\phi_1^{\pm}$$

$$- \frac{1}{2\pi}\int_0^{-2k_R} dl \int_{-\infty}^{\infty} d\xi \int_y^{\infty} d\eta\, e^{-il(\xi-x)+l(l+2k)(\eta-y)} q(\xi,\eta,t)\mu_1^{\pm},$$

and

$$\mu_2^{\pm} = 1 + \frac{1}{2\pi}\left(\int_{-\infty}^{-2k_R} dl + \int_0^{\infty} dl\right)\int_{-\infty}^{\infty} d\xi \int_0^y d\eta\, e^{-il(\xi-x)+l(l+2k)(\eta-y)} q(\xi,\eta,t)\mu_2^{\pm}$$

$$+ \frac{1}{2\pi}\left(\int_{-\infty}^{-2k_R} dl + \int_0^{\infty} dl\right)\int_{-\infty}^{\infty} d\xi \begin{cases} \int_0^t d\tau \\ -\int_t^T d\tau \end{cases} e^{-il(\xi-x)-l(l+2k)y-4il(l^2+3kl+3k^2)(\tau-t)} H(\xi,\tau,k,l)\phi_2^{\pm}$$

$$- \frac{1}{2\pi}\int_{-2k_R}^{0} dl \int_{-\infty}^{\infty} d\xi \int_y^{\infty} d\eta\, e^{-il(\xi-x)+l(l+2k)(\eta-y)} q(\xi,\eta,t)\mu_2^{\pm},$$

where

$$\phi_{1,2}^{\pm}(x,t,k_R,k_I) = \mu_{1,2}^{\pm}(x,0,t,k_R,k_I),$$

and

$$H(x,t,k,l) = 3\Big[g_x(x,t) - 2i(l+k)g(x,t) - \partial_x^{-1}h(x,t)\Big].$$

Sketch proof: These formulae are obtained by solving the direct problem:

1st Lax equation gives

$$\hat{\mu}(l, y, t, k_R, k_I) = \begin{cases} \int_0^y d\eta\, e^{l(l+2k)(\eta-y)} \widehat{q\mu}(l, \eta, t, k_R, k_I), & \\ +\hat{\mu}(l, 0, t, k_R, k_I)\, e^{-l(l+2k)y}, & l(l+2k_R) > 0, \\ & \\ -\int_y^\infty d\eta\, e^{l(l+2k)(\eta-y)} \widehat{q\mu}(l, \eta, t, k_R, k_I), & l(l+2k_R) < 0. \end{cases}$$

Note that the term $\hat{\mu}(l, 0, t, k_R, k_I)$ is new - not present in the ivp!

Inverting by means of

$$\begin{aligned} \mu(x, y, t, k_R, k_I) &= 1 + \frac{1}{2\pi} \int_{\mathbb{R}} e^{ilx} \hat{\mu}(l, y, t, k_R, k_I) dl \\ &= \frac{1}{2\pi} \int_{l(l+2k_R)>0} e^{ilx} \hat{\mu}(l, y, t, k_R, k_I) dl \\ &\quad + \frac{1}{2\pi} \int_{l(l+2k_R)<0} e^{ilx} \hat{\mu}(l, y, t, k_R, k_I) dl \end{aligned}$$

we find

$$\mu_1(x,y,t,k_R,k_I) =$$

$$= 1 + \frac{1}{2\pi}\left(\int_{-\infty}^{0} dl + \int_{-2k_R}^{\infty} dl\right) e^{ilx - l(l+2k)y} \int_0^y d\eta\, e^{l(l+2k)\eta}\widehat{q\mu}(l,\eta,t,k_R,k_I)$$

$$+ \frac{1}{2\pi}\left(\int_{-\infty}^{0} dl + \int_{-2k_R}^{\infty} dl\right) e^{ilx - l(l+2k)y}\hat{\phi}_1(l,t,k_R,k_I)$$

$$- \frac{1}{2\pi}\int_0^{-2k_R} dl\, e^{ilx - l(l+2k)y} \int_y^{\infty} d\eta\, e^{l(l+2k)\eta}\widehat{q\mu}(l,\eta,t,k_R,k_I), \quad k_R < 0,$$

and

$$\mu_2(x,y,t,k_R,k_I) =$$

$$= 1 + \frac{1}{2\pi}\left(\int_{-\infty}^{-2k_R} dl + \int_{0}^{\infty} dl\right) e^{ilx - l(l+2k)y} \int_0^y d\eta\, e^{l(l+2k)\eta}\widehat{q\mu}(l,\eta,t,k_R,k_I)$$

$$+ \frac{1}{2\pi}\left(\int_{-\infty}^{-2k_R} dl + \int_{0}^{\infty} dl\right) e^{ilx - l(l+2k)y}\hat{\phi}_2(l,t,k_R,k_I)$$

$$- \frac{1}{2\pi}\int_{-2k_R}^{0} dl\, e^{ilx - l(l+2k)y} \int_y^{\infty} d\eta\, e^{l(l+2k)\eta}\widehat{q\mu}(l,\eta,t,k_R,k_I), \quad k_R > 0.$$

How do we deal with $\hat{\phi}_{1,2}$?

First, note that evaluating at $y=0$ gives

$$\begin{aligned}\phi_1(x,t,k_R,k_I) = 1 + \frac{1}{2\pi}\left(\int_{-\infty}^{0} dl + \int_{-2k_R}^{\infty} dl\right) e^{ilx}\hat{\phi}_1(l,t,k_R,k_I)\\ - \frac{1}{2\pi}\int_0^{-2k_R} dl \int_0^{\infty} d\eta\, e^{ilx + l(l+2k)\eta}\widehat{q\mu}(l,\eta,t,k_R,k_I)\end{aligned}$$

hence

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty} dl\, e^{ilx}\hat{\phi}_1(l,t,k_R,k_I) = \frac{1}{2\pi}\left(\int_{-\infty}^{0} dl + \int_{-2k_R}^{\infty} dl\right) e^{ilx}\hat{\phi}_1(l,t,k_R,k_I)\\ - \frac{1}{2\pi}\int_0^{-2k_R} dl \int_0^{\infty} d\eta\, e^{ilx + l(l+2k)\eta}\widehat{q\mu}(l,\eta,t,k_R,k_I).\end{aligned}$$

Therefore, $\hat{\phi}_1$ does not satisfy any constraints for $l(l+2k_R) > 0$, whereas it does satisfy the following constraint for $l(l+2k_R) < 0$:

$$\hat{\phi}_1(l,t,k_R,k_I) = -\int_0^{\infty} d\eta\, e^{l(l+2k)\eta}\widehat{q\mu}(l,\eta,t,k_R,k_I), \quad l(l+2k_R) < 0.$$

Moreover, the second Lax equation evaluated at $y=0$ reads

$$\phi_t + 4\phi_{xxx} + 12ik\phi_{xx} - 12k^2\phi_x = F(x,0,t,k_R,k_I)\phi.$$

Applying the Fourier transform and integrating w.r.t t yields

$$\hat{\phi}(l,t,k_R,k_I) = \begin{cases} \int_0^t d\tau\, e^{-4il(l^2+3kl+3k^2)(\tau-t)}\,\widehat{H\phi} + e^{4il(l^2+3kl+3k^2)t}\,\hat{\phi}\big|_{t=0} \\ -\int_t^T d\tau e^{-4il(l^2+3kl+3k^2)(\tau-t)}\,\widehat{H\phi} + e^{-4il(l^2+3kl+3k^2)(T-t)}\hat{\phi}\big|_{t=T}, \quad k\in\mathbb{C}, \end{cases}$$

where

$$H(x,t,k,l) = 3\Big[g_x(x,t) - 2i(l+k)g(x,t) - \partial_x^{-1}h(x,t)\Big]$$

and

$$\widehat{H\phi}(l,t,k_R,k_I) = \int_{-\infty}^{\infty} dx\, e^{-ilx}\,(H\phi)\,(x,\tau,k_R,k_I).$$

Since

$$\mathrm{Re}\left[-4il(l^2+3kl+3k^2)\right] = 12k_I \cdot l\,(l+2k_R)\,,$$

requiring that ϕ is bounded forces us to consider the cases $k_I>0$ and $k_I<0$ (introducing μ^+ and μ^-) separately. After a few more manipulations, the direct problem is complete.

The global relation

Suppose that $\{q_0, g, h\}$ are given and let $\mu_0 = \mu(x, y, 0, k_R, k_I)$. By Green's theorem in the yt-plane, the Lax pair yields the identity:

$$\int_{-\infty}^{\infty} d\xi \int_0^{\infty} d\eta\, e^{-il\xi + l(l+2k)\eta} q_0(\xi,\eta)\mu_0 - \int_{-\infty}^{\infty} d\xi \int_0^{t} d\tau\, e^{-il\xi - 4il(l^2+3kl+3k^2)\tau} H(\xi,\tau,k,l)\phi$$
$$= \int_{-\infty}^{\infty} d\xi \int_0^{\infty} d\eta\, e^{-il\xi + l(l+2k)\eta - 4il(l^2+3kl+3k^2)t} q(\xi,\eta,t)\mu, \quad k \in \mathbb{C}, \quad l(l+2k_R) \leq 0.$$

Indeed, the first Lax equation gives

$$\left(\hat{\mu}\, e^{l(l+2k)y - 4il(l^2+3kl+3k^2)t}\right)_y = e^{l(l+2k)y - 4il(l^2+3kl+3k^2)t} \int_{-\infty}^{\infty} dx\, e^{-ilx} q\mu$$

while the second one implies

$$\left(\hat{\mu}\, e^{l(l+2k)y - 4il(l^2+3kl+3k^2)t}\right)_t = e^{l(l+2k)y - 4il(l^2+3kl+3k^2)t} \int_{-\infty}^{\infty} dx\, e^{-ilx} H\mu.$$

Thus

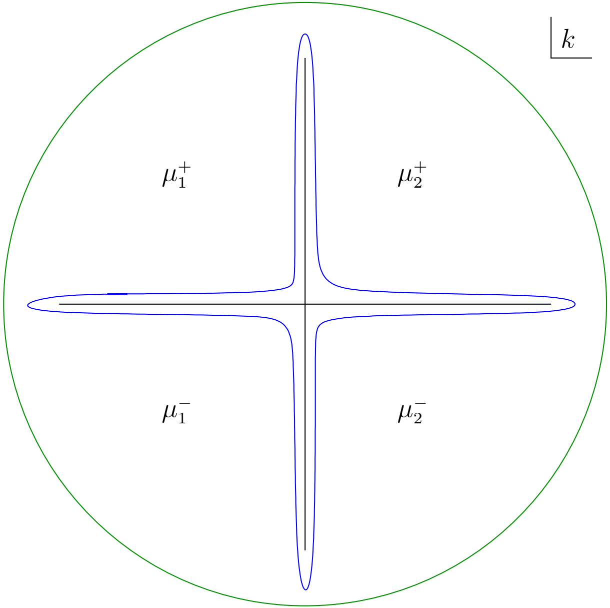
$$\left(e^{l(l+2k)y - 4il(l^2+3kl+3k^2)t} \int_{-\infty}^{\infty} dx\, e^{-ilx} q\mu\right)_t = \left(e^{l(l+2k)y - 4il(l^2+3kl+3k^2)t} \int_{-\infty}^{\infty} dx\, e^{-ilx} H\mu\right)_y.$$

Inverse problem

- Our aim is to use the formulae from the direct problem together with the global relation in order to reconstruct μ in terms of a spectral problem.
- For KPII, this turns out to be a $\bar{\partial}$ problem, which reflects that μ is a sectionally non-analytic function.
- Pompeiu's formula: For the closed boundary $\partial\mathcal{D}$ of the simple region $\mathcal{D}$, we have that if $\partial f/\partial\bar{z}$ is continuous then

$$\mu(k,\bar{k}) = \frac{1}{2i\pi}\int_{\partial\mathcal{D}} \frac{\mu(\nu,\bar{\nu})}{\nu-k}d\nu + \frac{1}{2i\pi}\int_{\partial\mathcal{D}} \frac{\partial\mu/\partial\bar{\nu}}{\nu-k}\, d\nu\wedge d\bar{\nu}.$$

- We will first express the $\bar{\partial}$ derivatives and the boundary values of μ as solutions of certain linear integral equations.
- Then, the above formula will be used to express μ in terms of another integral equation.



k
μ_1^+
μ_2^+
μ_1^-
μ_2^-

Pompeiu's formula in our case reads

$$
\begin{aligned}
\mu = 1 &+ \frac{1}{2i\pi}\int_{-\infty}^{0}\frac{d\nu_R}{\nu_R-k}\,\Delta\mu_1 + \frac{1}{2i\pi}\int_{0}^{\infty}\frac{d\nu_R}{\nu_R-k}\,\Delta\mu_2 \\
&+ \frac{1}{2\pi}\int_{0}^{\infty}\frac{d\nu_I}{i\nu_I-k}\,\delta\mu^+ + \frac{1}{2\pi}\int_{-\infty}^{0}\frac{d\nu_I}{i\nu_I-k}\,\delta\mu^- \\
&- \frac{1}{\pi}\int_{0}^{\infty}d\nu_R\int_{0}^{\infty}\frac{d\nu_I}{\nu-k}\,\frac{\partial\mu_2^+}{\partial\bar\nu} - \frac{1}{\pi}\int_{-\infty}^{0}d\nu_R\int_{0}^{\infty}\frac{d\nu_I}{\nu-k}\,\frac{\partial\mu_1^+}{\partial\bar\nu} \\
&- \frac{1}{\pi}\int_{0}^{\infty}d\nu_R\int_{-\infty}^{0}\frac{d\nu_I}{\nu-k}\,\frac{\partial\mu_2^-}{\partial\bar\nu} - \frac{1}{\pi}\int_{-\infty}^{0}d\nu_R\int_{-\infty}^{0}\frac{d\nu_I}{\nu-k}\,\frac{\partial\mu_1^-}{\partial\bar\nu},
\end{aligned}
$$

where

$$\Delta\mu_j(x,y,t,k_R) \doteq \left(\mu_j^+ - \mu_j^-\right)(x,y,t,k_R,0), \quad j=1,2.$$

and

$$\delta\mu^\pm(x,y,t,k_I) \doteq \left(\mu_1^\pm - \mu_2^\pm\right)(x,y,t,0,k_I).$$

- It actually turns out that $\delta\mu^\pm \equiv 0$.
- Deriving integral equations satisfied by the remaining jumps and $\bar\partial$ derivatives is certainly the hardest part of the problem.

The spectral map

1. $q_0 \mapsto \{\rho_1^+(x,y,k_R,k_I), \rho_2^+(x,y,k_R,k_I)\}$ for $k_I \geq 0$, where the functions ρ_1^+ and ρ_2^+ are defined in terms of q_0 via the following linear integral equations:

$$\rho_1^+ = 1 + \frac{1}{2\pi}\left(\int_{-\infty}^{0} dl + \int_{-2k_R}^{\infty} dl\right)\int_{-\infty}^{\infty} d\xi \int_0^y d\eta\, e^{-il(\xi-x)+l(l+2k)(\eta-y)} q_0 \rho_1^+$$
$$- \frac{1}{2\pi}\int_0^{-2k_R} dl \int_{-\infty}^{\infty} d\xi \int_y^{\infty} d\eta\, e^{-il(\xi-x)+l(l+2k)(\eta-y)} q_0 \rho_1^+, \quad k_R \leq 0,$$

$$\rho_2^+ = 1 + \frac{1}{2\pi}\left(\int_{-\infty}^{-2k_R} dl + \int_{0}^{\infty} dl\right)\int_{-\infty}^{\infty} d\xi \int_0^y d\eta\, e^{-il(\xi-x)+l(l+2k)(\eta-y)} q_0 \rho_2^+$$
$$- \frac{1}{2\pi}\int_{-2k_R}^{0} dl \int_{-\infty}^{\infty} d\xi \int_y^{\infty} d\eta\, e^{-il(\xi-x)+l(l+2k)(\eta-y)} q_0 \rho_2^+, \quad k_R \geq 0.$$

2. $\{q_0, g, h\} \mapsto \{\rho_1^-, \rho_2^-\}(x, y, k_R, k_I)$, $\{\phi_1^-, \phi_2^-\}(x, t, k_R, k_I)$ for and $k_I \le 0$, where the functions ρ_1^- and ρ_2^- are defined in terms of $\{q_0, g, h\}$ via the following linear integral equations:

$$
\begin{aligned}
\rho_1^- &= 1 + \frac{1}{2\pi}\left(\int_{-\infty}^{0} dl + \int_{-2k_R}^{\infty} dl\right)\int_{-\infty}^{\infty} d\xi \int_0^y d\eta\, e^{-il(\xi-x)+l(l+2k)(\eta-y)} q_0 \rho_1^- \\
&- \frac{1}{2\pi}\left(\int_{-\infty}^{0} dl + \int_{-2k_R}^{\infty} dl\right)\int_{-\infty}^{\infty} d\xi \int_0^T d\tau\, e^{-il(\xi-x)-l(l+2k)y-4il(l^2+3kl+3k^2)\tau} H\phi_1^- \\
&- \frac{1}{2\pi}\int_0^{-2k_R} dl \int_{-\infty}^{\infty} d\xi \int_y^{\infty} d\eta\, e^{-il(\xi-x)+l(l+2k)(\eta-y)} q_0 \rho_1^-, \quad k_R \le 0,
\end{aligned}
$$

$$
\begin{aligned}
\rho_2^- &= 1 + \frac{1}{2\pi}\left(\int_{-\infty}^{-2k_R} dl + \int_{0}^{\infty} dl\right)\int_{-\infty}^{\infty} d\xi \int_0^y d\eta\, e^{-il(\xi-x)+l(l+2k)(\eta-y)} q_0 \rho_2^- \\
&- \frac{1}{2\pi}\left(\int_{-\infty}^{-2k_R} dl + \int_{0}^{\infty} dl\right)\int_{-\infty}^{\infty} d\xi \int_0^T d\tau\, e^{-il(\xi-x)-l(l+2k)y-4il(l^2+3kl+3k^2)\tau} H\phi_2^- \\
&- \frac{1}{2\pi}\int_{-2k_R}^{0} dl \int_{-\infty}^{\infty} d\xi \int_y^{\infty} d\eta\, e^{-il(\xi-x)+l(l+2k)(\eta-y)} q_0 \rho_2^-, \quad k_R \ge 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\phi_1^- &= 1 - \frac{1}{2\pi}\left(\int_{-\infty}^{0} dl + \int_{-2k_R}^{\infty} dl\right)\int_{-\infty}^{\infty} d\xi \int_t^T d\tau e^{-il(\xi-x)-4il(l^2+3kl+3k^2)(\tau-t)} H\phi_1^- \\
&+ \frac{1}{2\pi}\int_0^{-2k_R} dl \int_{-\infty}^{\infty} d\xi \int_0^t d\tau e^{-il(\xi-x)-4il(l^2+3kl+3k^2)(\tau-t)} H\phi_1^-, \\
&- \frac{1}{2\pi}\int_0^{-2k_R} dl \int_{-\infty}^{\infty} d\xi \int_0^{\infty} d\eta\, e^{-il(\xi-x)+l(l+2k)\eta+4il(l^2+3kl+3k^2)t} q_0\rho_1^-,
\end{aligned}
$$

$$
\begin{aligned}
\phi_2^- &= 1 - \frac{1}{2\pi}\left(\int_{-\infty}^{-2k_R} dl + \int_{0}^{\infty} dl\right)\int_{-\infty}^{\infty} d\xi \int_t^T d\tau e^{-il(\xi-x)-4il(l^2+3kl+3k^2)(\tau-t)} H\phi_2^- \\
&+ \frac{1}{2\pi}\int_{-2k_R}^{0} dl \int_{-\infty}^{\infty} d\xi \int_0^t d\tau e^{-il(\xi-x)-4il(l^2+3kl+3k^2)(\tau-t)} H\phi_2^-, \\
&- \frac{1}{2\pi}\int_{-2k_R}^{0} dl \int_{-\infty}^{\infty} d\xi \int_0^{\infty} d\eta\, e^{-il(\xi-x)+l(l+2k)\eta+4il(l^2+3kl+3k^2)t} q_0\rho_2^-.
\end{aligned}
$$

3. $\{q_0, g, h\} \mapsto \{\alpha^-_{1,2}(k_R, k_I), \beta^{\pm}_{1,2}(k_R, k_I)\}$, where the functions $\alpha^-_{1,2}$ and $\beta^{\pm}_{1,2}$ are defined by the equations:

$$\alpha_1^-(k_R, k_I) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi \int_0^T d\tau\, e^{2ik_R\xi + 8ik_R(k_R^2 - 3k_I^2)\tau} H(\xi, \tau, k, -2k_R)\phi_1^-,$$

$$\alpha_2^-(k_R, k_I) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi \int_0^T d\tau\, e^{2ik_R\xi + 8ik_R(k_R^2 - 3k_I^2)\tau} H(\xi, \tau, k, -2k_R)\phi_2^-,$$

$$\beta_1^{\pm}(k_R, k_I) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi \int_0^{\infty} d\eta\, e^{2ik_R\xi - 4ik_Rk_I\eta} q_0\rho_1^{\pm},$$

$$\beta_2^{\pm}(k_R, k_I) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\xi \int_0^{\infty} d\eta\; e^{2ik_R\xi - 4ik_Rk_I\eta} q_0\rho_2^{\pm}.$$

4. $\{q_0, g, h\} \mapsto \{r_{1,2}(k_R, \lambda, l), p_1(k_R, l)\}$ with $l \in \mathbb{R}$, where the functions $r_{1,2}$ and p_1 are defined by the following equations:

$$r_2(k_R, \lambda, l) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi \int_0^{\infty} d\eta \, e^{-il\xi + l(l+2k_R)\eta} q_0(\xi, \eta) \left(e_\lambda \breve{\rho}_2^+ \right) (\xi, \eta, k_R, \lambda),$$

$$r_1(k_R, \lambda, l) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi \int_0^{\infty} d\eta \, e^{-il\xi + l(l+2k_R)\eta} q_0(\xi, \eta) \left(e_\lambda \breve{\rho}_1^+ \right) (\xi, \eta, k_R, \lambda),$$

$$p_1(k_R, l) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi \int_0^{T} d\tau \, e^{-il\xi - 4il(l^2 + 3k_R l + 3k_R^2)\tau} H(\xi, \tau, k_R, l) \phi_1^-(\xi, \tau, k_R, 0),$$

with

$$e_\lambda(x, y, t, k_R, \lambda) \doteq e^{-i(\lambda + 2k_R)x - \lambda(\lambda + 2k_R)y - 4i\lambda(\lambda^2 + 3k_R\lambda + 3k_R^2)t - 8ik_R^3 t}$$

and

$$\breve{\rho}_{1,2}^+(x, y, k_R, \lambda) = \rho_{1,2}^+(x, y, -k_R - \frac{\lambda}{2}, \frac{i\lambda}{2}).$$

5. $\{r_{1,2},\ p_1\} \mapsto \{\chi_{1,2}(k_R,\lambda), \psi_{1,2}(k_R,\lambda)\}$, where the functions $\chi_{1,2}$ and $\psi_{1,2}$ are defined via the following integral equations:

$$\chi_2(k_R,\lambda) - \int_{-\infty}^{\lambda} dl \chi_2(k_R,l) r_2(k_R,l,-2k_R-\lambda)$$

$$- \int_{-2k_R-\lambda}^{\infty} dl \chi_1(k_R,l) r_1(k_R,\lambda,-2k_R-\lambda) = p_1(k_R,-2k_R-\lambda), \quad k_R \leq 0,\ \lambda \leq 0,$$

$$\chi_1(k_R,\lambda) - \int_{-\infty}^{-2k_R-\lambda} dl \chi_2(k_R,l) r_2(k_R,l,-2k_R-\lambda)$$

$$- \int_{\lambda}^{\infty} dl \chi_1(k_R,l) r_1(k_R,l,-2k_R-\lambda) = p_1(k_R,-2k_R-\lambda), \quad k_R \leq 0,\ \lambda \geq -2k_R$$

and

$$\psi_2(k_R,\lambda) + \int_{-\infty}^{\lambda} dl \psi_2(k_R,l) r_2(k_R,l,-2k_R-\lambda)$$

$$+ \int_{0}^{-2k_R-\lambda} dl \psi_1(k_R,l) r_1(k_R,\lambda,-2k_R-\lambda) = p_1(k_R,-2k_R-\lambda),\ k_R \geq 0,\ \lambda \leq -2k_R,$$

$$\psi_1(k_R,\lambda) + \int_{-\infty}^{-2k_R-\lambda} dl \psi_2(k_R,l) r_2(k_R,l,-2k_R-\lambda)$$

$$+ \int_{0}^{\lambda} dl \psi_1(k_R,l) r_1(k_R,l,-2k_R-\lambda) = p_1(k_R,-2k_R-\lambda), \quad k_R \geq 0,\ \lambda \geq 0.$$

Inserting the spectral functions into Pompeiu's formula gives

$$\mu(x,y,t,k_R,k_I) =$$

$$1+\frac{1}{2i\pi}\int_{-\infty}^{0}\frac{d\nu_R}{\nu_R-k}\left[\int_{-\infty}^{0} d\lambda\,\chi_2\left(e_\lambda\breve{\mu}_2^+\right)+\int_{-2\nu_R}^{\infty} d\lambda\,\chi_1\left(e_\lambda\breve{\mu}_1^+\right)\right](x,y,t,\nu_R,\lambda)$$

$$+\frac{1}{2i\pi}\int_{0}^{\infty}\frac{d\nu_R}{\nu_R-k}\left[\int_{-\infty}^{-2\nu_R} d\lambda\,\psi_2\left(e_\lambda\breve{\mu}_2^+\right)+\int_{0}^{\infty} d\lambda\,\psi_1\left(e_\lambda\breve{\mu}_1^+\right)\right](x,y,t,\nu_R,\lambda)$$

$$+\frac{1}{\pi}\int_0^\infty d\nu_R\int_0^\infty\frac{d\nu_I}{\nu-k}\,\beta_2^+(\nu_R,\nu_I)E(x,y,t,\nu,-2\nu_R)\mu_1^+(x,y,t,-\nu_R,\nu_I)$$

$$-\frac{1}{\pi}\int_{-\infty}^0 d\nu_R\int_0^\infty\frac{d\nu_I}{\nu-k}\,\beta_1^+(\nu_R,\nu_I)E(x,y,t,\nu,-2\nu_R)\mu_2^+(x,y,t,-\nu_R,\nu_I)$$

$$+\frac{1}{\pi}\int_0^\infty d\nu_R\int_{-\infty}^0\frac{d\nu_I}{\nu-k}\,\left[\beta_2^- -\alpha_2^-\right](\nu_R,\nu_I)E(x,y,t,\nu,-2\nu_R)\mu_1^-(x,y,t,-\nu_R,\nu_I)$$

$$-\frac{1}{\pi}\int_{-\infty}^0 d\nu_R\int_{-\infty}^0\frac{d\nu_I}{\nu-k}\,\left[\beta_1^- -\alpha_1^-\right](\nu_R,\nu_I)E(x,y,t,\nu,-2\nu_R)\mu_2^-(x,y,t,-\nu_R,\nu_I)$$

where

$$E(x,y,t,k,l)\doteq e^{ilx-l(l+2k)y+4il(l^2+3kl+3k^2)t}.$$

Open problems

1. In the linear limit $q = \varepsilon u + O\left(\varepsilon^2\right),\ \varepsilon \to 0$, the KPII formulae yield the solution to the linearised KPII on the half-plane (as expected).

2. For a well posed problem, either g or h are prescribed as boundary conditions; on the other hand, the solution obtained depends on *both* g and h. Thus, in order for this solution to be effective it is necessary to use the global relation to eliminate the unknown boundary value. This problem remains open.

3. For the initial-value problem of the KPII, the linear integral equation analogous to the one coming from Pompeiu's formula admits a unique solution for μ for real initial conditions. This is due to the existence of a so-called "vanishing lemma", which is based on the theory of generalised analytic functions of Vekua. The question of whether there exists an analogous result in our case remains open.

4. The spectral functions are defined in terms of linear integral equations. The question of existence and uniqueness for these equations remains open.

5. In spite of the fact that the representation of q involves both g and h, it should still be possible to obtain effective formulae for the large t asymptotics of the solution. (Deift-Zhou in 2+1?)

6. In general, it is known that for the class of the so-called "linearisable" boundary conditions it is possible to express the spectral functions directly in terms of the given initial and boundary conditions using only *algebraic* manipulations. The question of identifying lineasable boundary conditions for the KPII remains open.

7. For KPII the formalism presented here involves the crucial assumption that several linear integral equations have a unique solution. In spite of the fact that these equations are of Fredholm type, it is not difficult to establish uniqueness under the assumption of sufficiently "small data". However, the elimination of the "small norm" assumption is a formidable task (see also Remark 4).

Thank you!