CATS COMPETITION[1]

University of Kentucky High School Math Day
October 2009

NO CALCULATORS, NO CELL PHONES!
WRITE YOUR ANSWERS IN THE PROVIDED BOXES

1 Express $0.12\bar{4}$ $(= 0.12444444\cdots)$ as a fraction in lowest terms.

Answer:

2 Given that
$$\frac{2}{x^2 - 4x + 3} = \frac{A}{x-1} + \frac{B}{x-3},$$
find A and B.

Answer: $A =$ $\qquad$ $B =$

3 A two-digit integer is equal to 3 times its ten's digit plus 5 times its one's digit. Find the number.

Answer:

4 What is the measure (in degrees) of the interior angle of a regular decagon (10 sides)?

Answer:

5 Compute
$$1 - \frac{1}{2} + \frac{3}{4} - \frac{7}{8} + \frac{15}{16} - \frac{31}{32} + \frac{63}{64} - \frac{127}{128} + \frac{255}{256}.$$

Answer:

6 Find all real solutions x to the equation
$$2^{x^2+2x} = 3^x.$$

Answer:

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[1]CATS stands for CATS Are Top Solvers.

7 The diameter of a quarter is approximately $15/16$ of an inch. How many non-overlapping quarters can one place inside a circle of diameter 3 inches?

Answer:

8 The perimeter of a triangle $\triangle ABC$ is 10 meters and the circumference of the inscribed circle of $\triangle ABC$ is 2 meters. Find the area of $\triangle ABC$.

Answer:

9 Compute
$$1776 + 1777 + 1778 + \cdots + 2008 + 2009.$$

Answer:

10 Suppose a fuel mixture is 4% ethanol and 96% gasoline. How much ethanol (in gallons) must you add to one gallon of fuel so that the new fuel mixture is 10% ethanol?

Answer:

11 Consider a 2009-gon $A_1 A_2 A_3 \ldots A_{2009}$. Starting at A_1, move to A_{22}, then to A_{43}, etc., jumping each time over 20 vertices. After how many jumps will you be back at A_1?

Answer:

12 Find all real solutions x to the equation
$$\mid 3 - |x - 1| \mid = 2$$
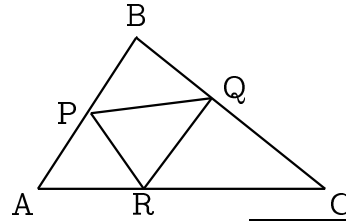and write your answers in increasing order.

Answer:

13 Two of the roots of the equation
$$x^3 + ax^2 + 7x + 3 = 0$$
are $x = -1$ and $x = -3$. What is the third root?

Answer:

14 The area of $\triangle ABC$ is 1. Point P is the midpoint of AB, and Q and R divide BC and AC so that $|BQ|/|QC| = 2/3$ and $|AR|/|RC| = 3/5$ as shown on the picture. Find the area of $\triangle PQR$.

Answer:

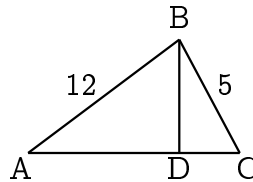**15** Solve the inequality
$$|5x - 7| < 5.$$

Answer:

16 Let
$$f(x) = \frac{1}{x} \quad \text{and} \quad g(x) = x + 1.$$
Find a positive solution x to the equation
$$g(f(g(f(x)))) = x.$$

Answer:

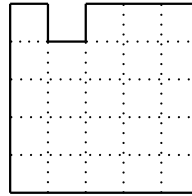**17** In $\triangle ABC$, $\angle B$ is a right angle, D is on AC and BD is perpendicular to AC. If $|AB| = 12$, $|BC| = 5$, find $|BD|, |AD|, |DC|$.

Answer: $|BD| =$ $, |AD| =$ $, |DC| =$

18 A 1×1 square is cut off a 5×5 square as shown in the picture. How many 2×1 dominoes can one put without overlapping or overhanging on what remains of the square?
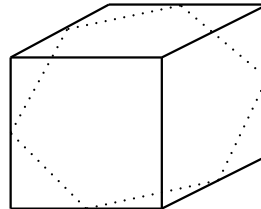
Answer:

19 If you expand
$$(1-x)(1-x^2)(1-x^4)(1-x^8)\cdots(1-x^{64}) = 1 - x - x^2 + x^3 - \cdots ,$$
what will the coefficient of x^{87} be?

Answer:

20 The volume of the cube is 1. The vertices of the hexagon are the midpoints of the edges of the cube. What is the area of the hexagon?
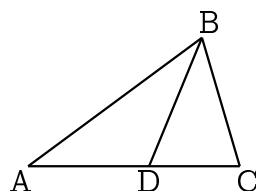
Answer:

21 On a straight North-South road you are allowed to move either 18 miles north or 25 miles south. After a number of such moves, will you be able to reach a point exactly 1 mile north of the starting point, and if so, what is the least number of moves it will take?

Answer:

22 In $\triangle ABC$, $\angle A$ and $\angle C$ measure $36°$ and $72°$ respectively. Point D is on AC and BD is the bisector of $\angle B$. If $|BC| = 3$, find $|AD| + |BD|$.

Answer:

23 An internet company coffee.moc sells two types of coffee, a "Traditional" type in 2 pound packs at \$15 per pack, and an "Organic" type in 3 pound packs at \$27 per pack. Yesterday, they sold 61 pounds of coffee for a total of \$507. How many packs of coffee did they sell yesterday?

Answer:

24 One glass is filled with 8 oz of milk. A second glass is filled with 8 oz of coffee. Suppose that an ounce of milk from the first glass is added to the coffee in the second glass, well mixed, and then an ounce of the mixture is added to the glass with milk.

(1) Will there be more milk in the coffee glass or more coffee in the milk glass?

Answer:

(2) What would happen if the contents of the second glass were not well mixed?
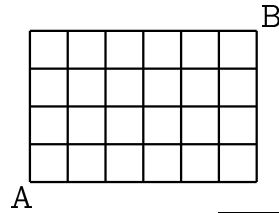
Answer:

25 Given
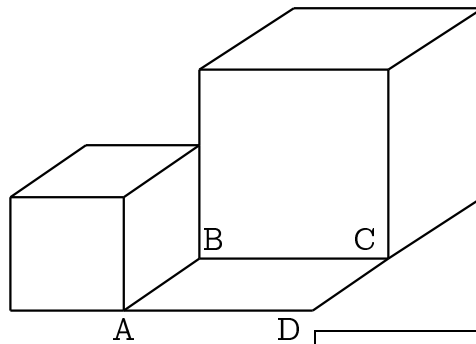$$(a^2 + b^2)^3 = (a^3 + b^3)^2,$$
find
$$\frac{a}{b} + \frac{b}{a}.$$

Answer:

26 A bug moves along a rectangular grid shown on the picture. If the bug can only move up or right, how many different routes are there for the bug to move from A to B?

B

A

Answer:

27 On the picture, one sees two cubes and a rectangle $ABCD$. If the rectangle has perimeter 10 and area 3, what is the combined volume of the two cubes?

B   C

A   D

Answer:

28 What digit is represented by the star in
$$2^{70} - 1 = 1180*91620717411303423?$$

Answer:

29 Which of the numbers 79^{80} or 80^{79} is larger?

Answer:

30 What integer is represented by the sum
$$\frac{1}{\log_2(100!)} + \frac{1}{\log_3(100!)} + \frac{1}{\log_4(100!)} + \frac{1}{\log_5(100!)} + \cdots + \frac{1}{\log_{100}(100!)}?$$

Answer: